The decomposition of global conformal invariants VI: The proof of the proposition on local Riemannian invariants.

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Abstract

This is the last in a series of papers where we prove a conjecture of Deser and Schwimmer regarding the algebraic structure of "global conformal invariants"; these are defined to be conformally invariant integrals of geometric scalars. The conjecture asserts that the integrand of any such integral can be expressed as a linear combination of a local conformal invariant, a divergence and of the Chern-Gauss-Bonnet integrand.

The present paper, jointly with [6, 7] gives a proof of an algebraic Proposition regarding local Riemannian invariants, which lies at the heart of our resolution of the Deser-Schwimmer conjecture. This algebraic Propositon may be of independent interest, applicable to related problems.

Contents

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1 Introduction

This paper is the sixth in the series of papers [3]–[8], where we confirm a conjecture of Deser and Schwimmer on the algebraic structure of "global conformal invariants". For the reader's convenience, we briefly review the Deser-Schwimmer conjecture:[1]

Definition 1.1 *Consider a Riemannian invariant $P(g)$ of weight $-n$ (n even). We will say that the integral $\int_{M^n} P(g)dV_g$ is a "global conformal invariant" if the value of $\int_{M^n} P(g)dV_g$ remains invariant under conformal re-scalings of the metric g.*

In other words, $\int_{M^n} P(g)dV_g$ is a "global conformal invariant" if for any $\phi \in C^\infty(M^n)$ we have $\int_{M^n} P(e^{2\phi}g)dV_{e^{2\phi}g} = \int_{M^n} P(g)dV_g$.

The Deser-Schwimmer conjecture [16] asserts:

Conjecture 1 *Let $P(g)$ be a Riemannian invariant of weight $-n$ such that the integral $\int_{M^n} P(g)dV_g$ is a global conformal invariant. Then there exists a local conformal invariant $W(g)$, a Riemannian vector field $T^i(g)$ and a constant $(Const)$ so that $P(g)$ can be expressed in the form:*

$$P(g) = W(g) + div_i T^i(g) + (Const) \cdot \text{Pfaff}(R_{ijkl}). \quad (1.1)$$

We prove:

Theorem 1.1 *Conjecture 1 is true.*

For the reader's convenience, we recall in brief the main results obtained in the earlier papers in this series; we then (again briefly) discuss the relationship of our work (and this paper in particular) with work related to Riemannian and conformal invariants, a subject largely inspired by Fefferman's program to understand the singularities in the Bergman and Szegö kernels of strictly

[1] We refer the reader to the introduction in [3] for a detailed discussion of the notions of "Riemannian invariant", "local conformal invariant".

pseudo-convex CR manifolds. Finally, we give a proper statement of the results we will be proving in the present paper and an outline of their proof.

In [3, 4, 5] we proved that the Deser-Schwimmer conjecture holds, *provided* one can show certain "Main algebraic propositions", namely Proposition 5.2 in [3] and Propositions 3.1, 3.2 in [4]. The next three papers, [6]–[8] (including the present one) are devoted to proving these "Main algebraic Propositons".

In [6] we set up a multiple induction by which we will prove these Propositions. In particular, we presented the "fundamemental Proposition" 2.1 in [6] (which we reproduce here, see Proposition 1.1 below) which is a generalization of the Main algebraic Propositions, and which depends on certain parameters (more on this below); we explained that we would prove Proposition 1.1 by an induction on these parameters. Since Proposition 2.1 is rather complicated to even write out, we first recall Proposition 5.2 in [3]:

A simplified description of the main algebraic Proposition 5.2 in [3]: Given a Riemannian metric g over an n-dimensional manifold M^n and auxilliary C^∞ scalar-valued functions $\Omega_1, \dots, \Omega_p$ defined over M^n, the objects of study are linear combinations of tensor fields $\sum_{l\in L} a_l C^{l,i_1\dots i_\alpha}_g$, where each $C^{l,i_1\dots i_\alpha}_g$ is a *partial contraction* with α free indices, in the form:

$$pcontr(\nabla^{(m)}R\otimes\dots\otimes\nabla^{(m_s)}R\otimes\nabla^{(b_1)}\Omega_1\otimes\dots\otimes\nabla^{(b_m)}\Omega_p); \tag{1.2}$$

here $\nabla^{(m)}R$ stands for the m^{th} covariant derivative of the curvature tesnor R,[2] and $\nabla^{(b)}\Omega_h$ stands for the b^{th} covariant derivative of the function Ω_h. A *partial contraction* means that we have list of pairs of indices $({}_a, {}_b), \dots, ({}_c, {}_d)$ in (1.2), which are contracted against each other using the metric g^{ij}. The remaining indices (which are not contracted against another index in (1.2)) are the *free indices* ${}_{i_1}, \dots, {}_{i_\alpha}$.

The "main algebraic Proposition" of [3] (roughly) asserts the following: Let $\sum_{l\in L_\mu} a_l C^{l,i_1\dots i_\mu}_g$ stand for a linear combination of partial contractions in the form (1.2), where each $C^{l,i_1\dots i_\mu}_g$ has a given number σ_1 of factors and a given number p of factor $\nabla^{(b)}\Omega_h$. Assume also that $\sigma_1+p\ge 3$, each $b_i\ge 2$,[3] and that for each pair of contracting indices $({}_a, {}_b)$ in any given $C^{l,i_1\dots i_\mu}_g$, the indices ${}_a, {}_b$ do not belong to the same factor. Assume also the rank $\mu>0$ is fixed and each partial contraction $C^{l,i_1\dots i_\mu}_g, l\in L_\mu$ has a given *weight* $-n+\mu$.[4] Let also $\sum_{l\in L_{>\mu}} a_l C^{l,i_1\dots i_{y_l}}_g$ stand for a (formal) linear combination of partial contractions of weight $-n+y_l$, with all the properties of the terms indexed in L_μ, *except* that now all the partial contractions have a different rank y_l, and each $y_l>\mu$.

The assumption of the "main algebraic Proposition" is a local equation in the form:

[2] In other words it is an $(m+4)$-tensor; if we write out its free indices it would be in the form $\nabla^{(m)}_{r_1\dots r_m}R_{ijkl}$.

[3] This means that each function Ω_h is differentiated at least twice.

[4] See [3] for a precise definition of weight.

$$\sum_{l\in L_\mu} a_l Xdiv_{i_1}\dots Xdiv_{i_\mu} C_g^{l,i_1\dots i_\mu} + \sum_{l\in L_{>\mu}} a_l Xdiv_{i_1}\dots Xdiv_{i_{y_l}} C_g^{l,i_1\dots i_{y_l}} = 0, \tag{1.3}$$

which is assumed to hold *modulo* complete contractions with $\sigma+1$ factors. Here given a partial contraction $C_g^{l,i_1\dots i_\alpha}$ in the form (1.2) $Xdiv_{i_s}[C_g^{l,i_1\dots i_\alpha}]$ stands for sum of $\sigma-1$ terms in $div_{i_s}[C_g^{l,i_1\dots i_\alpha}]$ where the derivative ∇^{i_s} is *not* allowed to hit the factor to which the free index ${}_{i_s}$ belongs.[5]

The main algebraic Proposition in [3] then claims that there will exist a linear combination of partial contactions in the form (1.2), $\sum_{h\in H} a_h C_g^{h,i_1\dots i_{\mu+1}}$ with all the properties of the terms indexed in $L_{>\mu}$, and all with rank $(\mu+1)$, so that:

$$\sum_{l\in L_\mu} a_l C_g^{l,(i_1\dots i_\mu)} + \sum_{h\in H} a_h Xdiv_{i_{\mu+1}} C_g^{l,(i_1\dots i_\mu)i_{\mu+1}} = 0; \tag{1.4}$$

the above holds modulo terms of length $\sigma+1$. Also the symbol $(\dots)$ means that we are *symmetrizing* over the indices between parentheses.

Local Invariants and Fefferman's program on the Bergman and Szegö kernels. The theory of *local* invariants of Riemannian structures (and indeed, of more general geometries, e.g. conformal, projective, or CR) has a long history. As stated above, the original foundations of this field were laid in the work of Hermann Weyl and Élie Cartan, see [25, 15]. The task of writing out local invariants of a given geometry is intimately connected with understanding polynomials in a space of tensors with given symmetries, which remain invariant under the action of a Lie group. In particular, the problem of writing down all local Riemannian invariants reduces to understanding the invariants of the orthogonal group.

In more recent times, a major program was laid out by C. Fefferman in [18] aimed at finding all scalar local invariants in CR geometry. This was motivated by the problem of understanding the local invariants which appear in the asymptotic expansions of the Bergman and Szegö kernels of strictly pseudo-convex CR manifolds, in a similar way to which Riemannian invariants appear in the asymptotic expansion of the heat kernel; the study of the local invariants in the singularities of these kernels led to important breakthroughs in [11] and more recently by Hirachi in [22]. It is worth noting that an analogous problem arises in the context of understanding the asymptotic expansion of the Szegö kernel of strictly pseudo-convex domains in $\mathbb{C}^n$ (or alternatively of abstract CR-manifolds). In particular, the leading term of the logarithmic singularity of the

[5] Recall that given a partial contraction $C_g^{l,i_1\dots i_\alpha}$ in the form (1.2) with σ factors, $div_{i_s} C_g^{l,i_1\dots i_\alpha}$ is a sum of σ partial contractions of rank $\alpha-1$. The first summand arises by adding a derivative ∇^{i_s} onto the first factor T_1 and then contracting the upper index ^{i_s} against the free index ${}_{i_s}$; the second summand arises by adding a derivative ∇^{i_s} onto the second factor T_2 and then contracting the upper index ^{i_s} against the free index ${}_{i_s}$ etc.

Szegö kernel exhibits a global invariance which is very similar to the one we discuss here, see [23].

This program was later extended to conformal geometry in [19]. Both these geometries belong to a broader class of structures, the *parabolic geometries*; these admit a principal bundle with structure group a parabolic subgroup P of a semi-simple Lie group G, and a Cartan connection on that principle bundle (see the introduction in [13]). An important question in the study of these structures is the problem of constructing all their local invariants, which can be thought of as the *natural, intrinsic* scalars of these structures.

In the context of conformal geometry, the first (modern) landmark in understanding *local conformal invariants* was the work of Fefferman and Graham in 1985 [19], where they introduced the *ambient metric*. This allows one to construct local conformal invariants of any order in odd dimensions, and up to order $\frac{n}{2}$ in even dimensions. The question is then whether *all* invariants arise via this construction.

The subsequent work of Bailey-Eastwood-Graham [11] proved that indeed in odd dimensions all conformal invariants arise via this construction; in even dimensions, they proved that the result holds when the weight (in absolute value) is bounded by the dimension. The ambient metric construction in even dimensions was recently extended by Graham-Hirachi, [21]; this enables them to indentify in a satisfactory way *all* local conformal invariants, even when the weight (in absolute value) exceeds the dimension.

An alternative construction of local conformal invariants can be obtained via the *tractor calculus* introduced by Bailey-Eastwood-Gover in [10]. This construction bears a strong resemblance to the Cartan conformal connection, and to the work of T.Y. Thomas in 1934, [24]. The tractor calculus has proven to be very universal; tractor bundles have been constructed [13] for an entire class of parabolic geometries. The relation betweeen the conformal tractor calculus and the Fefferman-Graham ambient metric has been elucidated in [14].

Broad discussion on the main algebraic Proposition: The present work, while pertaining to the questions above (given that it ultimately deals with the algebraic form of local *Riemannian* and *conformal* invariants[6]), nonetheless addresses a different *type* of problem: We here consider Riemannian invariants $P(g)$ for which the *integral* $\int_{M^n} P(g)dV_g$ remains invariant under conformal changes of the underlying metric; we then seek to understand the possible algebraic form of the *integrand* $P(g)$, ultimately proving that it can be de-composed in the way that Deser and Schwimmer asserted.

Now, Proposition 1.1[7] is purely a statement on local Riemannian invariants (in this case intrinsic scalar-valued functions which depend both on the metric and on certain auxilliary functions $\Omega_1, \dots, \Omega_p$). Thus, this is a proposition regarding the algebraic properties of the *classical* local Riemannian invariants.

[6]Indeed, the prior work on local *conformal* invariants played a central role in this endeavor, in [4, 5].

[7]We recall that this is a generalization of the "main algebraic Propositions" in [3, 4], which we outlined above.

While the author was led to led to the "main algebraic Propositions" out of the strategy that he felt was necessary to solve the Deser-Schwimmer conjecture, they can be thought of as results of independent interest. The *proof* of Proposition 1.1, presented in [6, 7, 7, 8] is in fact not particularily intuitive. It is proven by an induction (this is perhaps natural); the proof of the inductive step relies on studying the conformal variation of the assumption of Proposition 1.1 below. This is perhaps unexpected: Proposition 1.1 deals *purely* with Riemannian invariants; accordingly (1.6) holds for *all* Riemannian metrics. From that point of view, it is not obvious why restricting attention to the conformal variation of the equation (1.6) should provide useful information on the underlying algebraic form of the terms in (1.6). It is the author's sincere hope that deeper insight will be obtained in the future as to *why* the algebraic Propositions 5.2, 3.1,3.2 in [3, 4] hold.

Let us now recall Proposition 2.1 in [6]:

This claim (reproduced as Proposition 1.1 below) deals with tensor fields in the form:

$$\begin{aligned}
&pcontr(\nabla^{(m_1)}R_{ijkl}\otimes\dots\otimes\nabla^{(m_{\sigma_1})}R_{ijkl}\otimes\\
&S_*\nabla^{(\nu_1)}R_{ijkl}\otimes\dots\otimes S_*\nabla^{(\nu_t)}R_{ijkl}\otimes\\
&\nabla^{(b_1)}\Omega_1\otimes\dots\otimes\nabla^{(b_p)}\Omega_p\otimes\\
&\nabla\phi_{z_1}\dots\otimes\nabla\phi_{z_w}\otimes\nabla\phi'_{z_{w+1}}\otimes\dots\otimes\nabla\phi'_{z_{w+d}}\otimes\dots\otimes\nabla\tilde{\phi}_{z_{w+d+1}}\otimes\dots\otimes\nabla\tilde{\phi}_{z_{w+d+y}}).
\end{aligned} \tag{1.5}$$

(See the introduction in [6] for a detailed description of the above form). We recall that a (complete or partial) contraction in the above form is called "acceptable" $b_i\ge 2$ for every $1\le i\le p$. (In other words, we require that each of the functions Ω_i is differentiated at least twice).

The claim of Proposition 2.1 in [6] which we reproduce here is a generalization of the "main algebraic Propositions" in [3, 4]:

Proposition 1.1 *Consider two linear combinations of acceptable tensor fields in the form (1.5):*

$$\sum_{l\in L_\mu} a_l C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u),$$

$$\sum_{l\in L_{>\mu}} a_l C^{l,i_1\dots i_{\beta_l}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u),$$

where each tensor field above has real length $\sigma\ge 3$ and a given simple character $\vec{\kappa}_{simp}$. We assume that for each $l\in L_{>\mu}$, $\beta_l\ge\mu+1$. We also assume that none of the tensor fields of maximal refined double character in L_μ are "forbidden" (see Definition 2.12 in [6]).

We denote by

$$\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$$

a generic linear combination of complete contractions (not necessarily acceptable) in the form (1.8) below, that are simply subsequent to $\vec{\kappa}_{simp}$.[8] *We assume that:*

$$\begin{aligned}
&\sum_{l\in L_\mu} a_l Xdiv_{i_1}\dots Xdiv_{i_\mu} C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)+\\
&\sum_{l\in L_{>\mu}} a_l Xdiv_{i_1}\dots Xdiv_{i_{\beta_l}} C^{l,i_1\dots i_{\beta_l}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)+\\
&\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)=0.
\end{aligned}\tag{1.6}$$

We draw our conclusion with a little more notation: We break the index set L_μ *into subsets* $L^z, z\in Z$, *(Z is finite) with the rule that each* L^z *indexes tensor fields with the same refined double character, and conversely two tensor fields with the same refined double character must be indexed in the same* L^z. *For each index set* L^z, *we denote the refined double character in question by* $\vec{L}^z$. *Consider the subsets* L^z *that index the tensor fields of maximal refined double character.*[9] *We assume that the index set of those z's is* $Z_{Max}\subset Z$.

We claim that for each $z\in Z_{Max}$ *there is some linear combination of acceptable* $(\mu+1)$*-tensor fields,*

$$\sum_{r\in R^z} a_r C^{r,i_1\dots i_{\alpha+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u),$$

where each $C^{r,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ *has a* μ*-double character* $\vec{L^z_1}$ *and also the same set of factors* $S_*\nabla^{(\nu)}R_{ijkl}$ *as in* $\vec{L}^z$ *contain special free indices, so that:*

$$\begin{aligned}
&\sum_{l\in L^z} a_l C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon-\\
&\sum_{r\in R^z} a_r Xdiv_{i_{\mu+1}} C^{r,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon=\\
&\sum_{t\in T_1} a_t C^{t,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon,
\end{aligned}\tag{1.7}$$

modulo complete contractions of length $\ge\sigma+u+\mu+1$. *Here each*

$$C^{t,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$$

[8] Of course if $Def(\vec{\kappa}_{simp})=\emptyset$ then by definition $\sum_{j\in J}\dots=0$.

[9] Note that in any set S of μ-refined double characters with the same simple character there is going to be a subset S' consisting of the maximal refined double characters.

is acceptable and is either simply or doubly subsequent to $\vec{L}^z$.[10]

(See the first section in [6] for a description of the notions of *real length, acceptable tensor fields, simple character, refined double character, maximal refined double character, simply subsequent, strongly doubly subsequent.*

Proposition 1.1 is proven by an induction on four parameters, which we now recall:

The induction: Denote the left hand side of equation (1.6) by $L_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u)$ or just L_g for short. We recall that for the complete contractions in L_g, σ_1 stands for the number of factors $\nabla^{(m)} R_{ijkl}$ and σ_2 stands for the number of factors $S_* \nabla^{(\nu)} R_{ijkl}$. Also Φ stands for the total number of factors $\nabla\phi, \nabla\tilde{\phi}, \nabla\phi'$ and $-n$ stands for the weight of the complete contractions involved.

1. We assume that Proposition 1.1 is true for all linear combinations $L_{g^{n'}}$ with weight $-n'$, $n' < n$, n' even, that satisfy the hypotheses of our Proposition.

2. We assume that Proposition 1.1 is true for all linear combinations L_g of weight $-n$ and real length $\sigma' < \sigma$, that satisfy the hypotheses of our Proposition.

3. We assume that Proposition 1.1 is true for all linear combinations L_g of weight $-n$ and real length σ, with $\Phi' > \Phi$ factors $\nabla\phi, \nabla\tilde{\phi}, \nabla\phi'$, that satisfy the hypotheses of our Proposition.

4. We assume that Proposition 1.1 is true for all linear combinations L_g of weight $-n$ and real length σ, Φ factors $\nabla\phi, \nabla\tilde{\phi}, \nabla\phi'$ and with *fewer than $\sigma_1 + \sigma_2$ curvature factors* $\nabla^{(m)} R_{ijkl}, S_* \nabla^{(\nu)} R_{ijkl}$, provided L_g satisfies the hypotheses of our Proposition.

We will then prove Proposition 1.1 for the linear combinations L_g with weight $-n$, real length σ, Φ factors $\nabla\phi, \nabla\phi', \nabla\tilde{\phi}$ and with $\sigma_1 + \sigma_2$ curvature factors $\nabla^{(m)} R_{ijkl}$, $S_* \nabla^{(\nu)} R_{ijkl}$. So we are proving our Proposition by a multiple induction on the parameters $n, \sigma, \Phi, \sigma_1 + \sigma_2$ of the linear combination L_g.

In [6] we reduced the inductive step of Proposition 1.1 to three Lemmas 3.1, 3.2, 3.5;[11] (in particular we distinguished cases I,II,III on Proposition 1.1 by examiniming the tensor fields appearing in (1.6) and these three Lemmas corresponded to the three cases). In [6] and [7] we proved that these three Lemmas in [6], imply the inductive step of Proposition 1.1.

In the present paper we prove Lemmas 3.1, 3.2, 3,3, 3,4, 3.5. Lemmas 3.1, 3.2 in [6]. Lemmas 3.1, 3.2 will be derived in part A of the present paper, which

[10] Recall that "simply subsequent" means that the simple character of $C_g^{t,i_1 \ldots i_\mu}$ is subsequent to $Simp(\vec{L}^z)$.

[11] Lemma 3.5 in [6] depends on two *preparatory Lemmas*, 3.3, 3.4 in [6].

consists of sections 2, 3, 4. These two Lemmas are simpler to prove than Lemma 3.5; the analysis performed in part A will lay the groundwork for the proof of Lemma 3.5 in [6] (and Lemmas 3.3, 3.4 in [6]), in part B of the present paper, which consists of all the remaining sections.

For the reader's convenience, we will reproduce here the statements of Lemmas 3.1, 3.2, 3.5 from [6], which will be proven in the present paper.[12] There will be separate discussions in the beginning of parts A and B outlining the ideas and arguments that come into in the proofs of these Lemmas. For the reader's convenience, however, we will first provide a very simple sketch of the claims of these two Lemmas. We do this in order to present the *gist* of their claims, freed from the many notational conventions needed for the precise statement:

A simplified formulation of Lemmas 3.1, 3.2, 3.5 from [6]: The assumption of our Lemma is the equation (1.6). We recall that all the tensor fields appearing in that equation have the same *u-simple character*, which (in simple language) means that the factors $\nabla\phi_h$, $1 \le h \le u$ contract against the different factors $\nabla^{(m)}R_{ijkl}, S_*\nabla^{(\nu)}R_{jkl}, \nabla^{(A)}\Omega_h$ according to the *same pattern*; for example, if the factor $\nabla\phi_1$ contracts against the index ${}_i$ of a factor $S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ir_{\nu+1}kl}$ and the factor $\nabla\phi_4$ contracts against one of the indices ${}_{r_1},\dots,{}_{r_\nu},{}_{r_{\nu+1}}$ for a tensor field $C^{l_1,i_1\dots i_a}_g$ in (1.6), then the factors $\nabla\phi_1, \nabla\phi_4$ contract according to that rule in *all* the tensor fields in (1.6).

The notion of $Xdiv$: For a tensor field (i.e. a *partial contraction*) $C^{l,i_1\dots i_a}_g$ in the form (1.5), given a free index ${}_{i_s}$ which belongs to a factor T, the regular divergence $div_{i_s}C^{l,i_1\dots i_a}_g$ equals a sum of $\sigma+u$ $(a-1)$-tensor fields: The sum arises when we hit any of the $\sigma+u$ factors in $C^{l,i_1\dots i_a}_g$ by a derivative ∇^{i_s},[13] and then sum over all the resulting $(\mu-1)$-tensor fields. Now, $Xdiv_{i_s}C^{l,i_1\dots i_a}_g$ stands for the sum of $\sigma-1$ terms in the sum $div_{i_s}C^{l,i_1\dots i_a}_g$ where we *only* consider the $\sigma-1$ terms where the derivative ∇^{i_s} has hit a factor in one of the forms $\nabla^{(m)}R_{ijkl}, S_*\nabla^{(\nu)}R_{ijkl}$ or $\nabla^{(a)}\Omega_h$,[14] but *not* the factor T to which the free index ${}_{i_s}$ belongs. Thus, given any tensor field $C^{l,i_1\dots i_\alpha}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$, we can think of $Xdiv_{i_1}\dots Xdiv_{i_\alpha}C^{l,i_1\dots i_\alpha}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ as a linear combination of complete contractions in the form:

$$\begin{aligned}&pcontr(\nabla^{(m_1)}R_{ijkl}\otimes\dots\otimes\nabla^{(m_s)}R_{ijkl}\otimes\\&\nabla^{(b_1)}\Omega_1\otimes\dots\otimes\nabla^{(b_p)}\Omega_p\otimes\nabla\phi_1\otimes\dots\otimes\nabla\phi_u);\end{aligned} \tag{1.8}$$

The (simplified) statement of Lemma 1.1: This Lemma applies when there are tensor fields of rank μ in (1.6) with special free indices in factors $S_*\nabla^{(\nu)}R_{ijkl}$.[15] In that case, our Lemma picks out a particular subset of the tensor fields of rank μ, all of which have a special free index in a factor $S_*\nabla^{(\nu)}R_{ijkl}$

[12] These Lemmas are reproduced as Lemmas 1.1, 1.2, 1.3 in the present paper.

[13] (which contracts against the free index ${}_{i_s}$).

[14] In other words ∇^{i_s} is not allowed to hit one of the u factors $\nabla\phi_h$, $1 \le h \le \sigma$.

[15] Recall that a free index in a factor $S_*\nabla^{(\nu)}R_{ijkl}$ is *special* when it is one of the indices ${}_k,{}_l$.

(denote the index set of these tensor fields by $L^*_\mu \subset L_\mu$); it also picks out one of those special free indices–say the index ${}_{i_1}$, which will occupy the position ${}_k$ of the factor $S_*\nabla^{(\nu)}R_{ijkl}\nabla^i\tilde{\phi}_1$.

We then consider the $(\mu-1)$-tensor fields $C^{l,i_1\dots i_\mu}_g\nabla_{i_1}\phi_{u+1}$, $l \in L^*_\mu$.[16] The claim of Lemma 1.1 is (schematicaly) that there exists a linear combination of $(\mu+1)$-tensor fields, $\sum_{h\in H} a_h C^{h,i_1\dots i_{\mu+1}}_g \nabla_{i_1}\phi_{u+1}$, where each tensor field $C^{h,i_1\dots i_{\mu+1}}_g$ is a partial contraction in the form (1.5), with the same u-simple character $\vec{\kappa}_{simp}$, and with the index ${}_{i_1}$ occupying the position ${}_k$ in the factor $S_*\nabla^{(\nu)}R_{ijkl}$, such that:

$$\begin{aligned}
&\sum_{l\in L^*_\mu} a_l Xdiv_{i_2}\dots Xdiv_{i_\alpha} C^{l,i_1\dots i_\alpha}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1} = \\
&\sum_{h\in H} a_h Xdiv_{i_2}\dots Xdiv_{i_{\alpha+1}} C^{h,i_1\dots i_{\alpha+1}}_g \nabla_{i_1}\phi_{u+1} \\
&+\sum_{j\in J} a_j C^{j,i_1}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1};
\end{aligned} \tag{1.9}$$

here the terms indexed in J are "junk terms": they have length $\sigma+u$ (like the tensor felds indexed in L_1 and H) and are in the general form (1.8). They are "junk terms" because they have one of the two following features: either the index ${}_{i_1}$ (which contracts against the factor $\nabla\phi_{u+1}$) belongs to some factor $S_*\nabla^{(\nu)}R_{ijkl}$ but is *not* a special index, *or* one of the factors $\nabla\phi_h, 1 \le h \le u$) which are supposed to contract against the index ${}_i$ in some factor $S_*\nabla^{(\nu)}R_{ijkl}$ for $\vec{\kappa}_{simp}$ now contracts against a derivative index of some factor $\nabla^{(m)}R_{ijkl}$.[17]

The (simplified) statement of Lemma 1.2: This Lemma applies when no tensor fields of rank μ in (1.6) have special free indices in factors $S_*\nabla^{(\nu)}R_{ijkl}$,[18] but there are tensor fields of rank α that have special free indices in $\nabla^{(m)}R_{ijkl}$.[19] In that case, our Lemma picks out a particular subset of the tensor fields of rank μ, all of which have a special free index in a factor $\nabla^{(m)}R_{ijkl}$ (denote the index set of these tensor fields by $L^*_\mu \subset L_\mu$); it also picks out one of those special free indices–say the index ${}_{i_1}$, which will occupy the position ${}_i$ of the factor $\nabla^{(m)}R_{ijkl}$.

We then consider the $(\mu-1)$-tensor fields $C^{l,i_1\dots i_\mu}_g\nabla_{i_1}\phi_{u+1}$, $l \in L^*_\mu$.[20] The claim of Lemma 1.1 is (schematicaly) that there will exist a linear combination of $(\mu+1)$-tensor fields, $\sum_{h\in H} a_h C^{h,i_1\dots i_\mu}_g \nabla_{i_1}\phi_{u+1}$, where each tensor field

[16] These $(\mu-1)$-tensor fields arise from $C^{l,i_1\dots i_\mu}_g$ by just contracting the free index ${}_{i_1}$ against a new factor $\nabla\phi_{u+1}$.

[17] In the formal language of Lemma 1.1, introduced in [6], in this second scenario we would say that $C^{j,i_1}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ is "simply subequent' to the simple character $\vec{\kappa}_{simp}$.

[18] Recall that a free index in a factor $S_*\nabla^{(\nu)}R_{ijkl}$ is *special* when it is one of the indices ${}_k, {}_l$.

[19] Recall that a free index in a factor $\nabla^{(m)}R_{ijkl}$ is *special* when it is one of the indices ${}_i, {}_j, {}_k, {}_l$.

[20] These $(\mu-1)$-tensor fields arise from $C^{l,i_1\dots i_\mu}_g$ by just contracting the free index ${}_{i_1}$ against a new factor $\nabla\phi_{u+1}$.

$C_g^{h,i_1\dots i_{\mu+1}}$ is a partial contraction in the form (1.5), with the same u-simple character $\vec{\kappa}_{simp}$, and with the index ${}_{i_1}$ occupying the position ${}_i$ in the factor $\nabla^{(\nu)}R_{ijkl}$, such that:

$$\begin{aligned}
&\sum_{l\in L_\mu^*} a_l Xdiv_{i_2}\dots Xdiv_{i_\alpha} C_g^{l,i_1\dots i_\alpha}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1} = \\
&\sum_{h\in H} a_h Xdiv_{i_2}\dots Xdiv_{i_{\alpha+1}} C_g^{h,i_1\dots i_{\alpha+1}}\nabla_{i_1}\phi_{u+1} \\
&+\sum_{j\in J} a_j C_g^{j,i_1}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1};
\end{aligned} \tag{1.10}$$

here the terms indexed in J are "junk terms": they have length $\sigma+u$ (like the tensor felds indexed in L_1 and H) and are in the general form (1.8). They are "junk terms" because they have one of the two following features: either the index ${}_{i_1}$ (which contracts against the factor $\nabla\phi_{u+1}$) is a derivative index in some factor $\nabla^{(m)}R_{ijkl}$, *or* one of the factors $\nabla\phi_h, 1\le h\le u$ which are supposed to contract against the index ${}_i$ in some factor $S_*\nabla^{(\nu)}R_{ijkl}$ for $\vec{\kappa}_{simp}$ now contracts against a derivative index of some factor $\nabla^{(m)}R_{ijkl}$.[21]

The (simplified) statement of Lemma 1.3: Lemma 1.3 applies when all tensor fields of minimum rank μ in (1.6) have no special free indices in factors $S_*\nabla^{(\nu)}R_{ijkl}$ or $\nabla^{(m)}R_{ijkl}$.[22] In order to distinguish cases A and B of Lemma 1.3, we must recall some facts about the notion of *refined double character* of μ-tensor fields in the form (1.5) and *the maximal refined double character* among the μ-tensor fields appearing in (1.6):[23]

The notion of (refined) double character, and the comparison between different refined double characters: We recall that for a tensor field $C_g^{l,i_1\dots i_\mu}$ in the form (1.5) with no special free indices, its refined double character (which coincides with the *double character* in this case) encodes the pattern of distribution of the μ free indices among the different factors.

Furthermore, in [6] we introduced a weak ordering among refined double characters: Given two tensor fields $C_g^{l,i_1\dots i_\mu}, C_g^{r,i_1\dots i_\mu}$ (with the same simple character, say $\vec{\kappa}_{simp}$), we have introduced a comparison between their (refined) double characaters: We formed a list of the numbers of free indices that belong to the different factors, say $List_l=(s_1,\dots s_\sigma)$ and $List_r=(t_1,\dots t_\sigma)$, and considered the decreasing rearrangements of these lists, say $RList_l, RList_r$. We then decreed $C_g^{l,i_1\dots i_\mu}$ to be "doubly subsequent" to $C_g^{r,i_1\dots i_\mu}$ if $RList_r$ is lexicographically greater than $RList_l$. We also defined two *different* "refined

[21] In the formal language of Lemma 1.1, introduced in [6], in this second scenario we would say that $C_g^{j,i_1}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ is "simply subequent" to the simple character $\vec{\kappa}_{simp}$.

[22] Recall that a free index in a factor $S_*\nabla^{(\nu)}R_{ijkl}$ is *special* when it is one of the indices ${}_k,{}_l$; a free index in a factor $\nabla^{(m)}R_{ijkl}$ is *special* when it is one of the indices ${}_i,{}_j,{}_k,{}_l$.

[23] The reader is refered to [6] for precise definitions of these notions.

double characters" for which neither one is doubly subsequent to the other to be *equipolent*.

Now, cases A and B for Lemma 1.3 are distinuished as follows: Let us consider the different μ-tensor fields of maximal refined double character in (1.6). Let us suppose that their corresponding lists of distributions of free indices (in decreasing rearrangement),[24] is in the form $(M, s_1, \dots, s_{\sigma-1})$. Case A is when $s_1 \geq 2$. Case B is when $s_1 \leq 1$.

A rough description of Lemma 1.3 in case A: We canonicaly pick out a particular subset of the μ-tensor fields of maximal refined double character in (1.6); we denote the index set of these tensor fields by $L^z, z \in Z'_{Max}$ (μ-tensor fields with the same refined double character are indexed in the same index set L^z and vice versa).

For each $C^{l,i_1\dots i_\mu}_g$, $l \in \bigcup_{z\in Z'_{Max}} L^z$, we denote by $\dot{C}^{l,i_1\dots\hat{i}_{t\alpha+1}\dots i_\mu,i_*}_g$ the tensor field that arises from $C^{l,i_1\dots i_\mu}_g$ by erasing a certain particular index ${}_{i_{t\alpha+1}}$ and adding a free derivative index ${}_{i_*}$ onto a particular other factor(s).[25]

The claim of Lemma 1.3 is (schematicaly) that there will exist a linear combination of $(\mu+1)$-tensor fields, $\sum_{h\in H} a_h C^{h,i_1\dots i_{\mu+1}}_g \nabla_{i_1}\phi_{u+1}$, where each tensor field $C^{h,i_1\dots i_{\mu+1}}_g$ is a partial contraction in the form (1.5), with the same u-simple character $\vec{\kappa}_{simp}$, and with the index ${}_{i_1}$ being a non-special index in the crucial factor, such that:

$$\begin{aligned}&\sum_{z\in Z'_{Max}}\sum_{l\in L^z} a_l Xdiv_{i_2}\dots Xdiv_{i_\mu} C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1} = \\ &\sum_{l\in\tilde{L}} a_l Xdiv_{i_2}\dots Xdiv_{i_\mu} C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1} + \\ &\sum_{h\in H} a_h Xdiv_{i_2}\dots Xdiv_{i_{\mu+1}} C^{h,i_1\dots i_{\mu+1}}_g \nabla_{i_1}\phi_{u+1} \\ &+\sum_{j\in J} a_j C^{j,i_1}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}.\end{aligned} \tag{1.11}$$

Here the $(\mu-1)$-tensor fields indexed in $\tilde{L}$ are acceptable in the form (1.5) and also have length $\sigma+u$ (like the ones indexed in each L^z), but they are doubly subsequent to the $(\mu-1)$-tensor fields in the first line. The terms indexed in J are "junk terms"; they have length $\sigma+u$ (like the tensor felds indexed in L^z and H) and are in the general form (1.8). They are "junk terms" because one of the factors $\nabla\phi_h, 1 \leq h \leq u$) which are supposed to contract against the index ${}_i$ in some factor $S_*\nabla^{(\nu)}R_{ijkl}$ for the u-simple character $\vec{\kappa}_{simp}$ now contracts against

[24] As defined in the previous paragraph.

[25] As noted in [6], this operation is well-defined.

a derivative index of some factor $\nabla^{(m)}R_{ijkl}$.[26]

We note that we proved in [6] how Lemma 1.3 (in case A) implies the inductive step of Proposition 1.1.

A rough description of Lemma 1.3 in case B: In this case the claim of Lemma 3.5 in [6] coincides with that of Proposition 1.1.

The rigorous statement of Lemmas 3.1, 3.2 in [6]:

For both Lemmas 3.1, 3.2 in [6], we canonicaly pick out a particular subset of the μ-tensor fields of maximal refined double character in (1.6);[27] we denote the index set of these tensor fields by $L^z, z \in Z'_{Max}$ (μ-tensor fields with the same refined double character are indexed in the same index set L^z and vice versa).

Then, Lemma 3.1 in [6] asserts the following:

Lemma 1.1 *Assume (1.6), with weight $-n$, real length σ, $u = \Phi$ and $\sigma_1 + \sigma_2$ factors $\nabla^{(m)}R_{ijkl}, S_*\nabla^{(\nu)}R_{ijkl}$–assume also that the tensor fields of maximal refined double character are not "forbidden" (see Definition 2.12 in [6]). Suppose that there are μ-tensor fields in (1.6) with at least one special free index in a factor $S_*\nabla^{(\nu)}R_{ijkl}$. We then claim that there is a linear combination of acceptable tensor fields,*

$$\sum_{p\in P} a_p C^{p,i_1\dots i_b}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u),$$

each with $b \ge \mu+1$, with a simple character $\vec{\kappa}_{simp}$ and where each $C^{p,i_1\dots i_b}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ has the property that the free index ${}_{i_1}$ is the index ${}_k$ in the critical factor $S_*\nabla^{(\nu)}R_{ijkl}$ against which $\nabla\tilde{\phi}_{Min}$ is contracting, so that modulo complete contractions of length $\ge \sigma+u+2$:

$$\begin{aligned}
&\sum_{z\in Z'_{Max}}\sum_{l\in L^z} a_l Xdiv_{i_2}\dots Xdiv_{i_\mu} C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\
&\sum_{\nu\in N} a_\nu Xdiv_{i_2}\dots Xdiv_{i_\mu} C^{\nu,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}-\\
&\sum_{p\in P} a_p Xdiv_{i_2}\dots Xdiv_{i_b} C^{p,i_1\dots i_b}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1} =\\
&\sum_{t\in T} a_t C^{t,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}.
\end{aligned} \tag{1.12}$$

Here each $C^{\nu,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}$ is acceptable and has a simple character $\vec{\kappa}_{simp}$ (and ${}_{i_1}$ is again the index ${}_k$ in the critical factor $S_*\nabla^{(\nu)}R_{ijkl}$),

[26] In the formal language of Lemma 1.3, introduced in [6], we would say that $C^{j,i_1}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ is *simply subsequent* to the simple character $\vec{\kappa}_{simp}$.

[27] We refer the reader to the discussion above Proposition 2.1 in [6] for a rigorous definition of this notion.

but also has either strictly fewer than M free indices in the critical factor or is doubly subsequent to each $\vec{L}^z, z \in Z'_{Max}$. Each $C^{t,i_*}_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u)\nabla_{i_*}\phi_{u+1}$ is either simply subsequent to $\vec{\kappa}_{simp}$, or $C^{t,i_*}_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u)$ has a u-simple character $\vec{\kappa}_{simp}$ but the index ${}_{i_*}$ is not a special index. All complete contractions have the same weak $(u+1)$-simple character.

In order to state Lemma 3.2 in [6], we recall that it applies in the case where the tensor fields of maximal refined double character in (1.6) have special free indices in some factors $\nabla^{(m)}R_{ijkl}$, but no special free indices in any factor $S_*\nabla^{(\nu)}R_{ijkl}$. We recall also that for each $C^{l,i_1\ldots i_\mu}_g, l \in L^z, z \in Z'_{Max}$ the set $I_{*,l}$ stands for the index set of special free indices.

We also recall that for each $l \in L^z, z \in Z'_{Max}$ and each ${}_{i_h} \in I_{*,l}$ (we may assume with no loss of generality that ${}_{i_h}$ is the index ${}_i$ in some factor $\nabla^{(m)}R_{ijkl}$), we denote by $\tilde{C}^{l,i_1\ldots i_\mu}_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u)\nabla_{i_h}\phi_{u+1}$ the tensor field that arises from $C^{l,i_1\ldots i_\mu}_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u)\nabla_{i_h}\phi_{u+1}$ by replacing the expression $\nabla^{(m)}_{r_1\ldots r_m}R_{i_hjkl}\nabla^{i_h}\phi_{u+1}$ by an expression $S_*\nabla^{(m)}_{r_1\ldots r_m}R_{i_hjkl}\nabla^{i_h}\phi_{u+1}$. Then, Lemma 3.2 in [6] asserts:

Lemma 1.2 *Assume (1.6) with weight $-n$, real length σ, $u = \Phi$ and $\sigma_1 + \sigma_2$ factors $\nabla^{(m)}R_{ijkl}, S_*\nabla^{(\nu)}R_{ijkl}$. Suppose that no μ-tensor fields have special free indices in factors $S_*\nabla^{(\nu)}R_{ijkl}$, but some have special free indices in factors $\nabla^{(m)}R_{ijkl}$. In the notation above we claim that there exists a linear combination $\sum_{d\in D} a_d C^{d,i_1,\ldots,i_b}_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u, \phi_{u+1})$ of acceptable b-tensor fields (in the form (1.5) and $(u+1)$ factors $\nabla\phi$ and length $\sigma+u+1$) with a $(u+1)$-simple character $\vec{\kappa}'_{simp}$ and $b \geq \mu$, so that:*

$$
\begin{aligned}
&\sum_{z\in Z'_{Max}}\sum_{l\in L^z} a_l \sum_{i_h\in I_{*,l}} Xdiv_{i_1}\ldots X\hat{d}iv_{i_h}\ldots Xdiv_{i_\mu}\tilde{C}^{l,i_1\ldots i_\mu}_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) \\
&\nabla_{i_h}\phi_{u+1} + \sum_{\nu\in N} a_\nu Xdiv_{i_2}\ldots Xdiv_{i_\mu} C^{\nu,i_1\ldots i_\mu}_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u)\nabla_{i_1}\phi_{u+1} - \\
&\sum_{d\in D} a_d Xdiv_{i_1}\ldots Xdiv_{i_b} C^{d,i_1,\ldots,i_b}_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u, \phi_{u+1}) = \\
&\sum_{t\in T} a_t C^{t,i_*}_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u, \phi_u)\nabla_{i_*}\phi_{u+1},
\end{aligned}
\tag{1.13}
$$

where the $(\mu-1)$-tensor fields $C^{\nu,i_1\ldots i_\mu}_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u)\nabla_{i_1}\phi_{u+1}$ are acceptable, have $(u+1)$-simple character $\vec{\kappa}'_{simp}$ but also either have fewer than M free indices in the factor against which $\nabla_{i_h}\phi_{u+1}$ contracts,[28] or are doubly subsequent to all the refined double characters $\vec{\kappa}^z$, $z \in Z'_{Max}$. Moreover we require that each $C^{\nu,i_1\ldots,i_\mu}_g$ has the property that at least one of the indices ${}_{r_1}, \ldots, {}_{r_\nu}, {}_j$ in the factor $S_*\nabla^{(\nu)}_{r_1\ldots r_\nu}R_{ijkl}$ is neither free nor contracting against a factor $\nabla\phi'_h$,

[28] "Fewer than M free indices" where we also count the free index ${}_{i_h}$.

$h \leq u$. *The complete contractions* $C_g^{t,i_*}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_u)\nabla_{i_*}\phi_{u+1}$ *are simply subsequent to* $\vec{\kappa}'_{simp}$.

The rigorous statement of Lemma 3.5 in [6]:

Lemma 1.3 *Assume (1.6) with weight* $-n$, *real length* σ, $u = \Phi$ *and* $\sigma_1 + \sigma_2$ *factors* $\nabla^{(m)}R_{ijkl}, S_*\nabla^{(\nu)}R_{ijkl}$, *and additionally assume that no* μ*-tensor field in (1.6) has special free indices; assume also that* $L^*_\mu \bigcup L^+_\mu \bigcup L''_+ = \emptyset$ *(see the statement of Lemma 3.5 in [6] and the discussion above it). Recall the case A that we have distinguished above.*

Consider case A: Let k *stand for the (universal) number of second critical factors among the tensor fields indexed in* $\bigcup_{z\in Z'_{Max}} L^z$. *Let also* α *be the number of free indices in the (each) second critical factor in each* $C_g^{l,i_1\dots i_\mu}$, *for each* $z \in Z'_{Max}$. *We claim that:*

$$\begin{aligned}
&\binom{\alpha}{2}\sum_{z\in Z'_{Max}}\sum_{l\in L^z} a_l \sum_{r=0}^{k-1} Xdiv_{i_2}\dots Xdiv_{i_*}\dot{C}_g^{l,i_1\dots\hat{i}_{r\alpha+1}\dots i_\mu,i_*}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\\
&\nabla_{i_{r\alpha+2}}\phi_{u+1} + \sum_{\nu\in N} a_\nu Xdiv_{i_2}\dots Xdiv_{i_\mu} C_g^{\nu,i_1\dots,i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\
&\sum_{t\in T_1} a_t Xdiv_{i_1}\dots Xdiv_{i_{z_t}} C_g^{t,i_1\dots i_{z_t}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})+\\
&\sum_{t\in T_2} a_t Xdiv_{i_2}\dots Xdiv_{i_{z_t}} C_g^{t,i_1\dots i_{z_t}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\
&\sum_{t\in T_3} a_t Xdiv_{i_1}\dots Xdiv_{i_{z_t}} C_g^{t,i_1\dots i_{z_t}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})\\
&\big(+\sum_{t\in T_4} a_t Xdiv_{i_1}\dots Xdiv_{i_{z_t}} C_g^{t,i_1\dots i_{z_t}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})\big) =\\
&\sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}) = 0,
\end{aligned} \tag{1.14}$$

modulo complete contractions of length $\geq \sigma + u + 2$. *Here each* $C_g^{\nu,i_1\dots i_\mu}$ *is acceptable and has a simple character* $\vec{\kappa}^+_{simp}$ *and a double character that is doubly subsequent to each* $\vec{L}^{z,\sharp}, z \in Z'_{Max}$.[29]

$$\sum_{t\in T_1} a_t C_g^{t,i_1\dots i_{z_t}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$$

is a generic linear combination of acceptable tensor fields with a $(u+1)$*-simple character* $\vec{\kappa}^+_{simp}$*), and with* $z_t \geq \mu$.

[29] $\vec{L}^{z,\sharp}$ is the refined $(u+1,\mu-1)$-double character of the tensor fields $\dot{C}_g^{l,i_1\dots\hat{i}_{r\alpha+1}\dots i_\mu,i_*}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$, $z \in Z'_{Max}$.

$$\sum_{t\in T_2} a_t C_g^{t,i_1\dots i_{z_t}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$$

($z_t\ge\mu+1$) is a generic linear combination of acceptable tensor fields with a u-simple character $\vec{\kappa}_{simp}$, with the additional restriction that the free index $_{i_1}$ that belongs to the (a) crucial factor[30] *is a special free index.*[31]

Now, $t\in T_3$ means that there is one unacceptable factor $\nabla\Omega_h$ (and it is not contracting against any factor $\nabla\phi_t$) and moreover the tensor fields indexed in T_3 have $(u+1)$-simple character $\vec{\kappa}^+_{simp}$ and $z_t\ge\mu$.[32]

The sublinear combination $\sum_{t\in T_4}\dots$ appears only if the second critical factor is of the form $\nabla^{(B)}\Omega_k$, for some k. In that case, $t\in T_4$ means that there is one unacceptable factor $\nabla\Omega_k$, and it is contracting against a factor $\nabla\phi_r$: $\nabla_i\Omega_k\nabla^i\phi_r$, and moreover if $z_t=\mu$ then one of the free indices $_{i_1},\dots,_{i_\mu}$ is a derivative index, and moreover if it belongs to a factor $\nabla^{(B)}\Omega_h$ then $B\ge 3$.

Finally,

$$\sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$$

stands for a generic linear combination of complete contractions that are u-simply subsequent to $\vec{\kappa}_{simp}$.

In case B, we just claim that Proposition 1.1 is true.

1.1 Outline of Part A: The main strategy.

In part A of this paper we prove Lemmas 1.1 and 1.2, and set up the groundwork for the proof of Lemma 1.3 in part B.

The starting point of this proof will be the analysis of one local equation: We denote the assumption of these Lemmas (the equation (1.6)) by $L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)=0$, or just $L_g=0$ for short. The point of departure of our analysis will be to study the *first conformal variation* of this equation, $Image^1_{\phi_{u+1}}[L_g]=0$.[33]

Now, our first result here is to pick out a specific sublinear combination, $Image^{1,+}_{\phi_{u+1}}[L_g]$ in $Image^1_{\phi_{u+1}}[L_g]$ and to prove that it must vanish separately, modulo junk terms that we do not care about; this is the content of Lemma 2.1 and is done in subsection 2.2. Roughly speaking, the sublinear combination

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[30] I.e. the second critical factor, in this case.

[31] Recall that a special free index is either an index $_{k},_{l}$ in a factor $S_*\nabla^{(\nu)}R_{ijkl}$ or an internal index in a factor $\nabla^{(m)}R_{ijkl}$.

[32] If $z_t=\mu$ then we additionally claim that $\nabla\phi_{u+1}$ is contracting against a derivative index, and if it is contracting against a factor $\nabla^{(B)}\Omega_h$ then $B\ge 3$; moreover, in this case $C_g^{t,i_1\dots i_\mu}$ will contain no special free indices.

[33] Since the equation $L_g=0$ is assumed to hold for all Riemannian metrics g, we may consider its first variation under conformal deformations of g; i.e. $Image^1_{\phi_{u+1}}[L_g]=0$ is the new local equation $\frac{d}{dt}|_{t=0}L_{e^{2t\phi_{u+1}}g}=0$.

$Image^{1,+}_{\phi_{u+1}}[L_g]$ consists of the terms in $Image^{1}_{\phi_{u+1}}[L_g]$ which have one of two properties:

1. *Either* they have $\sigma+u+1$ factors (u of them in the form $\nabla\phi_1,\dots,\nabla\phi_u$ and a new one $\nabla\phi_{u+1}$),[34] have the u-weak character $Weak(\vec{\kappa}_{simp})$,[35] In rough terms, this means that the factors $\nabla\phi_1,\dots,\nabla\phi_u$ contract against the different factors $\nabla^{(m)}R_{ijkl},\nabla^{(p)}\Omega_h$ according to the same *pattern*, and also the new factor $\nabla\phi_{u+1}$ contracts against the "correct" factor.

2. *Or*, they have $\sigma+u$ factors, u of them in the form $\nabla\phi_1,\dots,\nabla\phi_u$, and a new one in the form $\nabla^{(A+2)}\phi_{u+1}$; this new factor has replaced one of the factors $\nabla^{(A)}R_{ijkl}$ or $S_*\nabla^{(A)}R_{ijkl}$ in L_g, by virtue of the transformation law (2.1). In this case, we additionnaly require that if we formally replace the factor $\nabla^{(A)}_{r_1\dots r_{A-2}r_{A-1}r_A}\phi_{u+1}$ by factor $\nabla^{(A-3)}_{r_1\dots r_{A-3}}R_{r_{A-2}r_{A-1}sr_A}\nabla^s\phi_{u+1}$ (by virtue of applying the curvature identity to the indices ${}_{r_{A-2}},{}_{r_{A-1}}$), then the resulting term would satisfy the first property above.

We next naturally break up $Image^{1,+}_{\phi_{u+1}}[L_g]$ into three sublinear combinations, see (6.1) below. In the rather technical subsection 2.3 we "get rid" of a specific sublinear combination in $Image^{1,+}_{\phi_{u+1}}[L_g]$ which would otherwise cause us trouble.

In section 3 we consider the terms in $Image^{1,+}_{\phi_{u+1}}[L_g]$ which have $\sigma+u$ factors. This sublinear combination is denoted by $CurvTrans[L_g]$. Our aim is to "get rid" of these terms (since the Lemmas we are proving assert claims about linear combinations with $\sigma+u+1$ factors), by introducing correction terms with $\sigma+u+1$ factors in total, *which we can control*.[36] In order to obtain correction terms which we can control, we argue as follows: We establish that the sublinear combination $CurvTrans[L_g]$ vanishes separately, modulo longer correction terms (which apriori we can *not control*). Moreover (as we check after multiple calculations in section 3.1), $CurvTrans[L_g]$ *retains* a lot of the algebraic structure of the terms in L_g; in particular, it can be expressed a linear combination of $Xdiv$'s of high order, plus terms which are "simply subsequent", in an appropriate sense. Thus, we iteratively apply *the inductive assumption* of Proposition 1.1, to derive that we can write $CurvTrans[L_g]=(Correction.Terms)$, where the correction terms have length $\sigma+u+1$ and *also* retain the algebraic structure that we want. We note that the analysis of section 3.1 applies to Lemmas 1.1, 1.2 *and* to Lemma 1.3.

Finally, in section 4 we deal directly with the terms in $Image^{1,+}_{\phi_{u+1}}[L_g]$ which are "born" with $\sigma+u+1$ factors. Our analysis in part A applies only to Lemmas 1.1, 1.2. After long calculations, we find that these terms have the algebraic features we would like them to; so if we add that sublinear combination to the correction terms we obtained from $CurvTrans[L_g]$, we straightforwardly derive our Lemmas 1.1, 1.2.

[34] Recall that the terms in L_g have $\sigma+u$ factors, u of them in the form $\nabla\phi_1,\dots,\nabla\phi_u$).

[35] This is the same u-weak character as for all terms in (1.6).

[36] This phrase *mostly* means that the corrections terms will be generic linear combinations which are allowed in the RHSs of the Lemmas we are proving.

A note is in order here: As explained in the simplified version of Lemmas 1.1, 1.2, terms with $\sigma + u + 1$ factors for which the factor $\nabla\phi_{u+1}$ contracts against a derivative index in a curvature factor $\nabla^{(m)} R_{ijkl}$ are considered "junk terms" in the statements of those Lemmas, and are thus allowed in the RHSs of our claims; we do not have to worry about their algebraic form. In other words, we are allowed to "throw away" many terms that apear in $Image^{1,+}_{\phi_{u+1}}[L_g]$, since tehy are allowed in the RHSs of the Lemmas 1.1, 1.2. This simplifies the task of proving the Lemmas 1.1, 1.2, and indeed we are able to derive them by this long analysis of the *single* equation $Image^{1,+}_{\phi_{u+1}}[L_g] = 0$. We will see in part B of this paper that the corresponding task for Lemma 3.5 in [6] will be more arduous.

Technical remark: In the setting of Lemma 1.2 there are certain special cases where the Lemma can not be derived from the analysis below; in those cases, Lemma 1.2 will be derived directly, in a "Mini-Appendix" at the end of pat A.[37] These special cases are when the tensor fieds of maximal refined double character in (1.6) satisfy:

1. Any factor $\nabla^{(m)} R_{ijkl}$ in one of the forms $\nabla^{(m)}_{r_1\dots r_m} R_{\sharp\sharp\sharp\sharp}$ or $\nabla^{(m)}_{r_1\dots r_m} R_{(free)\sharp\sharp\sharp}$, where each of the indices ${}_{r_1}, \dots, {}_{r_m}$ contracts against a factor $\nabla\phi_h$; each index ${}_\sharp$ is contracting against an index in another factor in $C^{l,i_1\dots i_\mu}_g$.

2. All the other factors in $C^{l,i_1\dots i_\mu}_g$ are simple factors in the form $S_* R_{ijkl}$, or factor $\nabla^{(2)}\Omega_h$ which are either simple and contain at most one free index or contract against exactly one $\nabla\phi_h$ and contain no free index.[38]

So when we deal with the Lemma 1.2 below, we will be assuming that the tensor fields of maximal refined double character in (1.6) are *not* in the forms described above.

2 Proof of Lemmas 1.1 and 1.2: Notation and preliminary results.

2.1 Codification of the assumptions:

Recall that the main assumption of Proposition 1.1 is the equation (1.6). This equation is also the main assumption for each of the Lemmas 1.1, 1.2 and Lemma 1.3. Recall that we are seeking to prove Lemmas 1.1, 1.2 and Lemma 1.3 for weight $-n$, where all the *tensor fields* in (1.6) have a given simple character $\vec{\kappa}_{simp}$ (with u factors $\nabla\phi$, and p factors $\nabla^{(y)}\Omega_h$).

In the remainder of this paper, we will prove Lemmas 1.1 and 1.2 and set the groundwork for the proof of Lemma 1.3 in part B.

[37]In fact, in that case the claims of Lemma 1.2 and Proposition 1.1 coincide.

[38]Recall that a factor $\nabla^{(A)}\Omega_h$ is called "simple" if it is not contracting against any factor $\nabla\phi_h$. A factor $S_*\nabla^{(\nu)} R_{ijkl}$ is called "simple" if its indices ${}_{r_1}, \dots, {}_{r_\nu}, {}_j$ are not contracting against any factor $\nabla\phi'_h$.

2.2 Introduction: Some technical tools.

The main tool in proving the Lemmas 1.1, 1.2 in the present paper, will be a careful analysis of one equation. This analysis will also be one of the main ingredients of the proof of Lemma 1.3 in part B of this paper. The starting point for this analysis will be the first conformal variation of the hypotheses of Lemmas 1.1 and 1.2. Recall that we denote the hypothesis of Lemmas 1.1 and 1.2 by $L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)=0$, for short. Then, the first conformal variation is the equation

$$Image^1_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=0,$$

where we recall that $Image^1_{\phi_{u+1}}[L_g]$ is defined via the formula $Image^1_{\phi_{u+1}}[L_g]=\frac{d}{dt}|_{t=0}L_{e^{2t\phi_{u+1}}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$. Given that L_g consists of complete contractions involving factors $\nabla^{(m)}R_{ijkl}$, $\nabla^{(p)}\Omega_h$, $\nabla\phi_h$, it will be useful to recall the transformation law (under the conformal change $\hat{g}_{ij}(x)=e^{2\phi(x)}g_{ij}(x)$) of the tensor R_{ijkl} and the Levi-Civita connection ∇:

$$\begin{aligned}
&R_{ijkl}(e^{2\phi}g)=e^{2\phi}[R_{ijkl}(g)+\nabla^{(2)}_{il}\phi g_{jk}+\nabla^{(2)}_{jk}\phi g_{il}-\nabla^{(2)}_{ik}\phi g_{jl}-\nabla^{(2)}_{jl}\phi g_{ik}\\
&+\nabla_i\phi\nabla_k\phi g_{jl}+\nabla_j\phi\nabla_l\phi g_{ik}-\nabla_i\phi\nabla_l\phi g_{jk}-\nabla_j\phi\nabla_k\phi g_{il}\\
&+|\nabla\phi|^2g_{il}g_{jk}-|\nabla\phi|^2g_{ik}g_{lj}],
\end{aligned}\tag{2.1}$$

$$\nabla_k\eta_l(e^{2\phi}g)=\nabla_k\eta_l(g)-\nabla_k\phi\eta_l-\nabla_l\phi\eta_k+\nabla^s\phi\eta_s g_{kl}.\tag{2.2}$$

Recall equation (1.6), which we re-write:

$$\begin{aligned}
&L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)=\sum_{l\in L_\mu}a_l Xdiv_{i_1}\dots Xdiv_{i_\mu}C^{l,i_1\dots,i_\mu}_g(\Omega_1,\dots,\Omega_p,\\
&\phi_1,\dots,\phi_u)+\sum_{l\in L'}a_l Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)+\\
&\sum_{j\in J\bigcup J^v}a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)=0;
\end{aligned}\tag{2.3}$$

the complete contractions indexed in J^v have length $\geq\sigma+u+1$ (the ones indexed in J have length $\sigma+u$). All *tensor fields* above are *acceptable*,[39] and have a given u-simple character $\vec{\kappa}_{simp}$.[40] The tensor fields indexed in L_μ have rank μ, and the ones indexed in L' have rank strictly greater than μ.[41] The complete contractions C^j are simply subsequent to $\vec{\kappa}_{simp}$. The above equation holds perfectly–not modulo longer complete contractions.

Clearly, the above equation implies that:

[39] Recall that this means that all functions Ω_h, $1\leq h\leq p$ are differentiated at least twice.
[40] See the definition 2.5 in [6] for the precise definition of this notion.
[41] The rank in question is not necessarily the same for each $l\in L'$.

$$Image^1_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] = 0. \tag{2.4}$$

In fact, we will not be using the equation (2.4) itself to prove Lemmas 1.1 and 1.2, but a slight variant of it:

We inquire which of the factors $\nabla^{(t_1)}\Omega_1,\dots,\nabla^{(t_p)}\Omega_p$ in $\vec{\kappa}_{simp}$ are not contracting against any factor $\nabla\phi_f$. With no loss of generality, we assume they are the factors $\nabla^{(t_1)}\Omega_1,\dots,\nabla^{(t_Y)}\Omega_Y$.

We will then be considering the equation:

$$\begin{aligned} &S_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}) = Image^1_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] \\ &+ L_g(\Omega_1\cdot\phi_{u+1},\Omega_2,\dots,\Omega_p,\phi_1,\dots,\phi_u) + \dots \\ &+ L_g(\Omega_1,\dots,\Omega_Y\cdot\phi_{u+1},\dots,\Omega_p,\phi_1,\dots,\phi_u) = 0, \end{aligned} \tag{2.5}$$

which holds perfectly, i.e. without correction terms.

Lemmas 1.1, 1.2 and 1.3 will be proven by carefully analyzing this equation. For now, we start by recalling some facts regarding "simple characters".

We recall that for all the tensor fields and complete contractions appearing in (2.3) and for each factor $\nabla\phi_f$, $f, 1\le f\le u$, there is a unique factor $\nabla^{(m)}R_{ijkl}$ or $S_*\nabla^{(\nu)}R_{ijkl}$ or $\nabla^{(B)}\Omega_h$ against which $\nabla\phi_f$ is contracting. Therefore, for each $f, 1\le f\le u$ we may unambiguously speak of *the* factor against which $\nabla\phi_f$ is contracting in each of the tensor fields and contractions in (1.6).

On the other hand, we may have factors $\nabla^{(m)}R_{ijkl}$ in $\vec{\kappa}_{simp}$ that are not contracting against any factor $\nabla\phi_h$. We recall that there is the same number of such factors for all tensor fields with the given simple character $\vec{\kappa}_{simp}$. We will sometimes refer to such factors as "generic factors of the form $\nabla^{(m)}R_{ijkl}$".

Notational conventions: Examine the conclusions of Lemmas 1.1, 1.2. Focus on the first lines of those conclusions. For Lemma 1.2 all the tensor fields in the first line have the same $(u+1)$-simple character which we will denote by $\vec{\kappa}^+_{simp}$. For Lemma 1.1 we observe that all tensor fields in the first line of the conclusion have the same u-simple character $\vec{\kappa}_{simp}$, and furthermore the factor $\nabla\phi_{u+1}$ is contracting against the index $_k$ in a specified factor $T = S_*\nabla^{(\nu)}R_{ijkl}$ in $\vec{\kappa}_{simp}$. While strictly speaking these tensor fields are not in the form (1.5),[42] and hence we cannot speak of a simple character, we will abuse language and define that in the context of the proof of Lemma 1.1, any tensor field that is contracting against $(u+1)$ factors $\nabla\phi_h$ has a $(u+1)$-simple character $\vec{\kappa}^+_{simp}$ if it satisfies the properties explained in the previous sentence.

Now, recall that there is a well-defined notion of the "crucial factor(s)" in the context of Lemmas 1.1, 1.2, introduced in [6]. In particular, for each of the three Lemmas above, we may unambiguously speak of the set of factors $\nabla\phi_h$ in $\vec{\kappa}_{simp}$ that are contracting against the crucial factor. (Recall that this set may also be empty–in that case we have a *set* of crucial factors, which are all the factors $\nabla^{(m)}R_{ijkl}$ that are not contracting against any $\nabla\phi_h$).

[42] Because the factor $\nabla\phi_{u+1}$ is contracting against a special index in a factor S_*R_{ijkl}.

We denote by $(\vec{\kappa}_{simp})_1$ the set of numbers h for which $\nabla\phi_h$ is contracting against the crucial factor.

We will now introduce some further notation, for future reference. We will formally construct a new u-simple character *for contractions with* $\sigma+u$ *factors* (whereas $\vec{\kappa}^+_{simp}$ corresponds to contractions with $\sigma+u+1$ factors in total). (The definition that follows is highly un-intuitive, but the discussion below provides some intuition).

Definition 2.1 *Consider the simple character $\vec{\kappa}_{simp}$. Pick a tensor field (in the form (1.5)) with a u-simple character $\vec{\kappa}_{simp}$. In the case where the crucial factor(s) is (are) of the form $S_*\nabla^{(\nu)}R_{ijkl}$ or $\nabla^{(m)}R_{ijkl}$, we will define a new u-simple character $pre\vec{\kappa}^+_{simp}$ as follows:*

Recall that $\vec{\kappa}_{simp}$ is a list of sets. Consider the entry in $\vec{\kappa}_{simp}$ that corresponds to the crucial factor. That entry will either be a set of numbers S_h or a set in the form $(\{\alpha\}, S_h)$, where α is a number and S_h a set of numbers. Respectively, the entry will either belong to the list L_1 or the list L_2. Then $pre\vec{\kappa}^+_{simp}$ arises from the simple character $\vec{\kappa}_{simp}$ by erasing this entry $(\{\alpha\}, S_h)$ or S_h from L_1 or L_2 and adding an entry $S_{p+1} = S_h\bigcup\{\alpha\}$ or $S_{p+1} = S_h$, respectively, in L_3.

A more intuitive description of $pre\vec{\kappa}^+_{simp}$ is the following: Consider any complete contraction C_g with a u-simple character $\vec{\kappa}_{simp}$. Consider a crucial factor T in C_g,[43] along with its indices, $T_{r_1\dots r_{m+4}}$. Assume that two of the indices in T (say ${}_{r_{m+2}}, {}_{r_{m+4}}$) are not contracting against factors $\nabla\phi$ (this can always be done, i.e. we can always find a contraction C_g that satisfies this requirement-by adding derivative indices onto T if necessary).

Formally replace $T_{r_1\dots r_{m+4}}$ by a new factor $\nabla^{(m+4)}_{r_1\dots r_{m+1}r_{m+3}}\Omega_{p+1}$. The indices that contracted against ${}_{r_{m+2}}, {}_{r_{m+4}}$ now become free. $pre\vec{\kappa}^+_{simp}$ is then the u-simple character of this new partial contraction.

Now, in the setting of Lemma 1.3 we will slightly generalize the above notions. Although we have defined a notion of "crucial factor" (which is either the "critical" or the "second critical" factor), we wish to allow ourselves some extra freedom, and thus we will be picking another factor (set of factors) in $\vec{\kappa}_{simp}$ and we will call it (them) the *selected factor(s)*.

Our choice of selected factor(s) is entirely free: We can either pick any well-defined factor in $\vec{\kappa}_{simp}$ and call it the selected factor (recall a *well-defined* factor is either a curvature factor that is contracting against some $\nabla\phi$ or a factor $\nabla^{(p)}\Omega_h$), or we can pick the *set* of factors $\nabla^{(m)}R_{ijkl}$ in $\vec{\kappa}_{simp}$ that are not contracting against any factors $\nabla\phi$ and call all those factors the *selected factors*.

Having chosen a (set of) selected factor(s) in $\vec{\kappa}_{simp}$, say F_s, we then formally construct the $(u+1)$-simple character $\vec{\kappa}^+_{simp}$ by just adding a derivative ∇_a onto the (one of the) factor(s) F_s and just contracting ${}_a$ against a new factor $\nabla^a\phi_{u+1}$ (and S_*-symmetrizing if F_s is of the form $S_*\nabla^{(\nu)}R_{ijkl}$). Then, if F_s is a curvature factor, we define the $(u+1)$-simple character $pre\vec{\kappa}^+_{simp}$ as in Definition 2.1.

[43] For this definition the crucial factor must be a curvature factor.

(If F_s is not a curvature term, then $pre\vec{\kappa}^+_{simp}$ is undefined).

A note: In the rest of this section, we will be referring exclusively to the "selected" factor. In the setting of Lemmas 1.1, 1.2 and in case A of Lemma 1.3, we will take the selected factor(s) to be the crucial factor(s). Therefore, in most circumstances the two notions coincide.

Now, we define sublinear combinations in $Image^1_{\phi_{u+1}}[L_g]$, where L_g is the left hand side of our hypothesis (2.3). We note that these definitions still make sense for *any* linear combination L_g consisting of complete contractions with a given weak character $\vec{\kappa}$ and a given (set of) selected factor(s) T in $\vec{\kappa}$.

Definition 2.2 *We denote by* $Image^{1,+}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ *the sublinear combination in* $Image^1_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ *that consists of complete contractions* $C_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ *with the following properties:*

1. $C_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})$ *must have no internal contractions.*
2. *If* $C_g(\Omega_1,\dots,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})$ *has length* $\sigma+u+1$ *then it must have a factor* $\nabla\phi_{u+1}$ *(with only one derivative) and a weak character* $Weak(\vec{\kappa}^+_{simp})$.
3. *If* $C_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})$ *has length* $\sigma+u$, *then it has a factor* $\nabla^{(A)}\phi_{u+1}$, $A\ge 2$. *Moreover, it must have a weak u-character* $Weak(pre\vec{\kappa}^+_{simp})$.

Definition 2.3 *Define* $Image^{1,\alpha}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ *to be the sublinear combination in* $Image^1_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ *that consists of complete contractions* $C_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ *with length* $\sigma+u+1$ *and a factor* $\nabla^{(A)}\phi_{u+1}$ *with* $A\ge 2$. *We also denote a generic linear combination of such complete contractions by* $\sum_{z\in Z}a_zC^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$.

We define $Image^{1,\beta}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ *to stand for the sublinear combination of complete contractions* $C_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$ *in* $Image^1_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ *with the following properties:*

1. $C_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$ *must have precisely one internal contraction.*
2. *Either* $C_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$ *has length* $\sigma+u+1$ *and a factor* $\nabla\phi_{u+1}$.
3. *Or* $C_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$ *has length* $\sigma+u$ *and a factor* $\nabla^{(A)}\phi_{u+1}$ *(*$A\ge 2$*) (which does not contain the internal contraction).*

Finally, we define $Image^{1,\gamma}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ *to stand for the sublinear combination of complete contractions* $C_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$ *in* $Image^1_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ *with one of the properties:*

1. *Either* $C_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$ *has length* $\sigma+u$, *a factor* $\nabla^{(A)}\phi_{u+1}$ *and a weak character which is not* $Weak(pre\vec{\kappa}^+_{simp})$,

2. *Or* $C_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$ *has length* $\sigma+u+1$ *and a factor* $\nabla\phi_{u+1}$ *but its weak character is not* $Weak(\vec{\kappa}^+_{simp})$.

We make note of the fact that we are *not* imposing any restriction to the weak character of the complete contractions that belong to $Image^{1,\beta}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$.

Now, assuming any equation of the form $L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)=0$ (this equation is assumed to hold perfectly–here L_g consists of complete contractions with length $\geq\sigma+u$) we observe that:

$$Image^{1}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=Image^{1,+}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+$$
$$Image^{1,\alpha}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+Image^{1,\beta}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+$$
$$Image^{1,\gamma}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)](=0), \tag{2.6}$$

(modulo complete contractions of length $\geq\sigma+u+2$).

This follows by the definitions above and the transformation laws (2.1) and (2.2).

Our next claim will be used frequently in the future, so we present it in somewhat general notation:

Lemma 2.1 *Assume that* $L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ *is a linear combination of complete contractions with no internal contractions and with a given weak character* $Weak(\vec{\kappa})$ *(where* $\vec{\kappa}$ *is any chosen simple character consisting of* $\sigma+u$ *factors). Assume that*

$$L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)=0, \tag{2.7}$$

modulo complete contractions with length $\geq\sigma+u+1$. *Then, in the notation of Definition 2.2, we claim:*

$$Image^{1,+}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\sum_{z\in Z}a_zC^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})=0, \tag{2.8}$$

modulo complete contractions of length $\geq\sigma+u+2$. *Here* $\sum_{z\in Z}$ *stands for a generic linear combination of contractions with* $\sigma+u+1$ *factors, one of which is in the form* $\nabla^{(c)}\phi_{u+1}$, $c\geq 2$.

Proof of Lemma 2.1: Our point of departure is equation (2.6). In view of that equation, we notice that it would suffice to show that:

$$Image^{1,\beta}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\sum_{z\in Z}a_zC^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),\tag{2.9}$$

$$Image^{1,\gamma}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\sum_{z\in Z}a_zC^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),\tag{2.10}$$

modulo complete contractions of length $\geq \sigma+u+2$. If we can show the above equations, then by virtue of (2.6) we will immediately deduce our claim.

We will denote by $Image^{1,\beta,\sigma+u}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$, $Image^{1,\gamma,\sigma+u}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ the sublinear combinations of complete contractions with length $\sigma+u$ in $Image^{1,\beta}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$, $Image^{1,\gamma}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$, respectively. We will also denote by $\sum_{w\in W^\beta}a_wC^w_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$, $\sum_{w\in W^\gamma}a_wC^w_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ generic linear combinations of complete contractions as the ones that belong to the linear combinations $Image^{1,\beta}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$, $Image^{1,\gamma}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$, respectively, which in addition have length $\sigma+u+1$.

We claim that:

$$Image^{1,\beta,\sigma+u}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\sum_{w\in W^\beta}a_wC^w_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$$
$$+\sum_{z\in Z}a_zC^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),\tag{2.11}$$

$$Image^{1,\gamma,\sigma+u}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\sum_{w\in W^\gamma}a_wC^w_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$$
$$+\sum_{z\in Z}a_zC^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).\tag{2.12}$$

These equations are proven by the usual argument: Recall that (2.6) holds formally. Then, we observe that if we pick out the sublinear combinations in those equations that consist of complete contractions with $\sigma+u$ factors (denote those sublinear combinations by Z_g), then $lin\{Z_g\}=0$ formally. (Recall that $lin\{Z_g\}$ stands for the *linearization* of the linear combination Z_g–see [3]). Now, since both the weak character and the number of internal contractions are *invariant* under the permutations that make $lin\{Z_g\}$ formally zero, we derive that the linearizations of the left hand sides of (2.11), (2.12) must vanish formally.

Hence, by repeating the permutations in the non-linear setting, we obtain the right hand sides in (2.11), (2.12) as correction terms.

Thus, substituting the above two equations into (2.6), we obtain an equation in place of (2.6), where the sums analogous to $Image^{1,\beta}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$, $Image^{1,\gamma}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ contain no complete contractions of length $\sigma+u$. So if we denote by $Image^{1,+,\sigma+u}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ the sublinear combination in $Image^{1,+}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ that consists of complete contractions with $\sigma+u$ factors, we obtain that $Image^{1,+,\sigma+u}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=0$, modulo longer complete contractions. But then, since this equation must hold formally at the linearized level, by just repeating the permutations that make the linearized linear combination formally zero we derive, modulo complete contractions of length $\geq \sigma+u+2$:

$$Image^{1,+,\sigma+u}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] = \sum_{h\in H} a_h C^h_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$$
$$+\sum_{z\in Z} a_z C^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}), \tag{2.13}$$

where each $C^h_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$ has length $\sigma+u+1$ and a factor $\nabla\phi_{u+1}$ and no internal contractions and also has a weak character $Weak(\vec{\kappa}^+)$.

By replacing this also into (2.6), we derive:

$$\sum_{h\in H} a_h C^h_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}) + \sum_{w\in W^\beta} a_w C^w_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}) +$$
$$\sum_{w\in W^\gamma} a_w C^w_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}) + \sum_{z\in Z} a_z C^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}) = 0. \tag{2.14}$$

Now, since the above must hold formally (and the weak character as well as the number of internal contractions are invariant under the permutations that make the linearizations of the left hand side formally zero), we derive that:

$$\sum_{w\in W^\beta} a_w C^w_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}) = 0,$$
$$\sum_{w\in W^\gamma} a_w C^w_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}) = 0,$$

modulo complete contractions of length $\geq \sigma+u+2$.

Thus, (2.11), (2.12) combined with the above two equation complete the proof of Lemma 2.1. $\Box$

Further break-up of $Image^{1,+}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$:

We break
$$Image^{1,+}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$
into three sublinear combinations: We define $CurvTrans[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ to stand for the sublinear combination that arises by applying the transformation law (2.1) to a factor $\nabla^{(m)}R_{ijkl}$ or $S_*\nabla^{(\nu)}R_{ijkl}$, in some complete contraction in L_g. We observe that the complete contractions in $CurvTrans[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ have length $\sigma+u$. They will each be in the form:

$$contr(\nabla^{(m_1)}R_{ijkl}\otimes\dots\otimes\nabla^{(m_s)}R_{ijkl}\otimes S_*\nabla^{(\nu_1)}R_{ijkl}\otimes\dots\otimes S_*\nabla^{(\nu_\tau)}R_{ijkl}$$
$$\nabla^{(m)}\phi_{u+1}\otimes\nabla^{(f_1)}\Omega_1\otimes\dots\otimes\nabla^{(f_p)}\Omega_p\otimes\nabla\phi_1\otimes\dots\otimes\nabla\phi_u). \tag{2.15}$$

We denote by $LC[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ the sublinear combination in $Image^{1,+}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ that arises by applying the transformation law (2.2). Finally, we denote by $W[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ the sublinear combination in $Image^{1,+}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ that arises by applying the transformation $R_{ijkl}\longrightarrow e^{2\phi_{u+1}}R_{ijkl}$ and bringing out an expression $\nabla\phi_{u+1}$.

Then, by definition:

$$Image^{1,+}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=CurvTrans[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$
$$+LC[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+W[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]. \tag{2.16}$$

Much of this section and of the next ones consists of understanding the three sublinear combinations above and of using our inductive assumption on Corollary 1 in [3] in order to derive our three Lemmas 1.1, 1.2, 1.3.

2.3 Preliminary Work.

We will be generically denoting all the tensor fields that appear in (1.6) by $C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$. Also, $C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ will stand for a generic complete contraction with a weak character $Weak(\vec{\kappa}_{simp})$, where C^j_g is simply subsequent to $\vec{\kappa}_{simp}$.

We need a definition in order to formulate our claim:

Definition 2.4 *We define $LC_\Phi[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ and $LC_\Phi[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ to stand for the sublinear combinations in $LC[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$, $LC[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$,*[44]

[44] Recall that by definition these sublinear combinations consist of complete contractions of length $\sigma+u+1$ with weak character $Weak(\vec{\kappa}^+_{simp})$.

that arise when we bring out a factor $\nabla\phi_{u+1}$ by applying the transformation law (2.2) to a pair of indices $(\nabla_A, {}_B)$ where ∇_A denotes a derivative index, and where either ∇_A or ${}_B$ is contracting against a factor $\nabla\phi_h$ $(1 \le h \le u)$.

We observe that a complete contraction in $LC_\Phi[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$, $LC_\Phi[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ can only arise by applying the transformation law (2.2) to a pair of indices $(\nabla_A, {}_B)$ in one of the following two ways:

Firstly, if the index ∇_A is contracting against a factor $\nabla\phi_h$ and the index ${}_B$ is contracting against the selected factor and we bring out the third summand in (2.2). Alternatively, if ${}_B$ is contracting against a factor $\nabla\phi_h$ and ∇_A is contracting against the selected factor and we bring out the second summand in (2.2).

The aim of this subsection is to show the following:

Lemma 2.2 *Consider (1.6). Then, in the notation of definition 2.4, we claim:*

$$\begin{aligned}&\sum_{l\in L}a_lLC_\Phi[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\&\sum_{j\in J}a_jLC_\Phi[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\&\sum_{l\in L'}a_lXdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\&\sum_{j\in J}a_jC^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),\end{aligned}\tag{2.17}$$

where $\sum_{l\in L'}a_lC^{l,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}$, in the setting of Lemmas 1.1 and 1.2, stands for a generic linear combination of acceptable tensor fields of length $\sigma+u+1$, with $a\ge\mu$ and with a $(u+1)$-simple character $\vec{\kappa}^+_{simp}$. On the other hand, in the setting of Lemma 1.3 it stands for a generic linear combination in the form

$$\sum_{t\in T_1\bigcup T_2}a_tC^{t,i_1\dots i_{z_t}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})\nabla_{i_1}\phi_{u+1}.$$

$\sum_{j\in J}a_jC^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$ stands for a generic linear combination of complete contractions that are simply subsequent to $\vec{\kappa}^+_{simp}$ (in the setting of all three Lemmas).

Proof of Lemma 2.2:

In order to show this Lemma, we will again introduce some preliminary definitions. We recall that in equation (2.3) all the complete contractions of

length $\sigma + u$ have the same weak character.[45] For each partial contraction with a simple character $\vec{\kappa}_{simp}$, we let $\{F_1, \dots, F_X\}$ stand for the set of *non-selected* factors that are contracting against some factor $\nabla\phi_f$.

Then, for each $1 \le h \le X$, we define an operation $Hit^h_{\nabla\tau}$ that formally acts on each tensor field $C^{l,i_1\dots i_a}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)$ and each complete contraction $C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)$ in (1.6) by hitting the factor F_h by a derivative index ∇_u and then contracting $_u$ against a factor $\nabla_u\tau$. We also define:

$$\begin{aligned}&Hit_{\nabla\tau}[Xdiv_{i_1} \dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)] = \\ &\sum_{h=1}^{X} Hit^h_{\nabla\tau}[Xdiv_{i_1} \dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)],\end{aligned} \tag{2.18}$$

and also:

$$Hit_{\nabla\tau}[C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)] = \sum_{h=1}^{X} Hit^h_{\nabla\tau}[C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)]. \tag{2.19}$$

Since (2.3) holds formally, we derive that:

$$\begin{aligned}&\sum_{l\in L} a_l Hit_{\nabla\tau}[Xdiv_{i_1} \dots Xdiv_{i_\alpha} C^{l,i_1\dots i_a}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)] + \\ &\sum_{j\in J} a_j Hit_{\nabla\tau}[C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)] = 0,\end{aligned} \tag{2.20}$$

modulo complete contractions of length $\ge \sigma + u + 2$.

Terminology: For any complete contraction C_g in the form (1.5) or (1.8) with a weak character $Weak(\vec{\kappa}_{simp})$, we have assigned symbols $F_1, \dots, F_X$ to their factors $\nabla^{(m)}R_{ijkl}, S_*\nabla^{(\nu)}R_{ijkl}, \nabla^{(p)}\Omega_h$ in C_g which are contracting against some $\nabla\phi_h$. Now, regarding the complete contractions in $Image^1_{\phi_{u+1}}[C_g]$, we impose the following conventions:

For each contraction of length $\sigma + u + 1$ in $Image^{1,+}_{\phi_{u+1}}[C_g]$ we will still speak of the factors $F_1, \dots, F_X$. (They can be identified, since all the complete contractions in $Image^{1,+}_{\phi_{u+1}}[C_g]$ still have a u-simple character $\vec{\kappa}_{simp}$). On the other hand, for each contraction of length $\sigma+u$ in $Image^{1,+}_{\phi_{u+1}}[C_g]$ (with a factor $\nabla^{(v+2)}\phi_{u+1}$ that has arisen from some $F_a = \nabla^{(m)}R_{ijkl}$ or $F_a = S_*\nabla^{(\nu)}R_{ijkl}$), we will speak of the factors $F_1, \dots, F_X$, only now F_a will be the factor $\nabla^{(v+2)}\phi_{u+1}$.

We now separately consider the sublinear combinations in $Image^1_{\phi_{u+1}}[Xdiv_{i_1} \dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)]$ and $Image^1_{\phi_{u+1}}[C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)]$ that belong to $Image^{1,+}_{\phi_{u+1}}[L_g]$. We denote these sublinear combinations by

[45] See [6] for a precise definition of this notion.

$Image^{1,+}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$,
$Image^{1,+}_{\phi_{u+1}}[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$. Then, (2.8) can be re-expressed as:

$$\sum_{l\in L}a_l Image^{1,+}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+ \\ \sum_{j\in J}a_j Image^{1,+}_{\phi_{u+1}}[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\sum_{z\in Z}a_z C^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u), \tag{2.21}$$

modulo complete contractions of length $\ge\sigma+u+2$.

Thus, we again define the operation $Hit_{\nabla\tau}$ on the linear combinations

$$Image^{1,+}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)],$$

$Image^{1,+}_{\phi_{u+1}}[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ as in (2.18), (2.19).

On the other hand, we can also consider $Image^{1}_{\phi_{u+1}}\{Hit_{\nabla\tau}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}$ and define $Image^{1,+}_{\phi_{u+1}}\{Hit_{\nabla\tau}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}$ in the same way as above.

This leads us to consider the quantity:

$$Difference[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]= \\ Hit_{\nabla\tau}\{Image^{1,+}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}- \\ Image^{1,+}_{\phi_{u+1}}\{Hit_{\nabla\tau}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}. \tag{2.22}$$

Analogously, we define a quantity:

$$Difference[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)],$$

for each $j\in J$.

Then, since (2.3) holds formally, and just by virtue of the transformation laws (2.1), (2.2) and our definitions above we have that:

$$\sum_{l\in L}a_l Difference[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+ \\ \sum_{j\in J}a_j Difference[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=0 \tag{2.23}$$

modulo complete contractions of length $\ge\sigma+u+3$. This equation, suitably analyzed, will imply Lemma 2.2.

Easily, we observe that both
$Difference[Xdiv_{i_1}\dots Xdiv_{i_a}C_g^{l,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ and
$Difference[C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ consist of complete contractions with length $\geq \sigma+u+2$, each with one factor $\nabla\phi_{u+1}$ and one factor $\nabla\tau$. This is because the contractions with length $\sigma+u+1$ in (2.22) will cancel perfectly (without correction terms with more factors).

Now, we write out:

$$\begin{aligned}&\sum_{l\in L} a_l Difference[Xdiv_{i_1}\dots Xdiv_{i_a}C_g^{l,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] =\\ &\sum_{t\in T} a_t C_g^t(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\tau),\end{aligned} \tag{2.24}$$

$$\begin{aligned}&\sum_{j\in J} a_j Difference[C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] =\\ &\sum_{y\in Y} a_y C_g^y(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\tau).\end{aligned} \tag{2.25}$$

We then consider the sets of complete contractions C^t and C^y for which the factor $\nabla\tau$ is contracting against a given factor F_h, $1\leq h\leq X$.[46] We denote the index sets of those complete contractions by Y^h, T^h. Of course, since (2.23) holds formally, it follows that for each $h\leq X$:

$$\begin{aligned}&\sum_{t\in T^h} a_t C_g^t(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\tau)+\\ &\sum_{y\in Y^h} a_y C_g^y(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\tau)=0,\end{aligned} \tag{2.26}$$

modulo complete contractions of length $\geq\sigma+u+2$.

Now, for each $h\in\{1,\dots,x\}$ we define $Set(h)=\{a_1,\dots,a_{r_h}\}$ as follows: $\rho\in Set(h)$ if and only if $\nabla\phi_\rho$ is contracting against F_h in $\vec{\kappa}_{simp}$. For each $\rho\in Set(h)$, we define $G^{\rho,\tau}[C_g^t(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\tau)]$, $G^{\rho,\tau}[C_g^y(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\tau)]$ to stand for the complete contraction that arises by replacing the factors $\nabla_i\tau$, $\nabla_j\phi_\rho$ by a factor g_{ij} (we thus obtain a complete contraction with length $\sigma+u$ and one internal contraction). We also define:

[46] Recall that $\{F_h\}_{1\leq h\leq X}$ is the set of all the non-selected factors that are contracting against some factor $\nabla\phi_v, v\leq u$.

$$G\{\sum_{l\in L} a_l Difference[Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+$$
$$\sum_{j\in J} a_j Difference[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\} =$$
$$\sum_{h=1}^{X}\sum_{t\in T^h} a_t \sum_{\rho\in Set(h)} G^{\rho,\tau}\{C^t_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\tau)\}+$$
$$\sum_{h=1}^{X}\sum_{y\in Y^h} a_y \sum_{\rho\in Set(h)} G^{\rho,\tau}\{C^y_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\tau)\}. \tag{2.27}$$

Then, since (2.23) holds formally, we derive that modulo complete contractions of length $\geq \sigma+u+1$:

$$\sum_{h=1}^{X}\sum_{t\in T^h} a_t \sum_{\rho\in Set(h)} G^{\rho,\tau}\{C^t_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\tau)\}+$$
$$\sum_{h=1}^{X}\sum_{y\in Y^h} a_y \sum_{\rho\in Set(h)} G^{\rho,\tau}\{C^y_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\tau)\} = 0. \tag{2.28}$$

Now, we wish to explicitly understand how the left hand side of (2.28) arises from $\sum_{l\in L} a_l Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$, $\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$. We will find that the left hand sides of these equations are closely related to the LHS of the equation in Lemma 2.2 (which we are trying to prove).

We introduce some notation in order to accomplish this.

Definition 2.5 *We define an operation $G^\sharp$ that formally acts on the complete contractions in the linear combinations*

$$LC_\Phi[Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)], LC_\Phi[C^f_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]:$$

Firstly, we define an operation $G^{\sharp,h,a_b}$ for each $h, 1\leq h\leq X$ and each $a_b\in Set(h)=\{a_1,\dots,a_{r_h}\}$. $G^{\sharp,h,a_b}$ formally acts by erasing the factor $\nabla\phi_{a_b}$ that contracts against the factor F_h (say against an index $_\chi$) and then by hitting the factor F_h by a derivative ∇^χ (so we again obtain a complete contraction with an internal contraction and with length $\sigma+u$). Then, we define:

$$G^\sharp\{LC_\Phi[Xdiv_{i_1}\dots Xdiv_{i_a}C_g^{l,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}$$
$$=\sum_{h=1}^{X}\sum_{a_b\in Set(h)=\{a_1,\dots,a_{r_h}\}}G^{\sharp,h,a_b}\{LC_\Phi[Xdiv_{i_1}\dots Xdiv_{i_a} \tag{2.29}$$
$$C_g^{l,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\},$$

$$G^\sharp\{LC_\Phi[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}$$
$$=\sum_{h=1}^{X}\sum_{a_b\in Set(h)=\{a_1,\dots,a_{r_h}\}}G^{\sharp,h,a_b}\{LC_\Phi[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}. \tag{2.30}$$

Now, for our next definition we will treat each factor $S_*\nabla^{(\nu)}R_{ijkl}$ as a sum of factors in the form $\nabla^{(\nu)}R_{ijkl}$. We do this for the purpose of picking out the factors $\nabla\phi_h$ that are contracting against internal indices in each of the summands $\nabla^{(\nu)}R_{ijkl}$ of the un-symmetrization of $S_*\nabla^{(\nu)}R_{ijkl}$. Thus, we are now considering tensor fields and complete contractions in the form (1.8) in paper [5]. For each tensor field $C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ and also for each complete contraction $C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ we consider the factors $\nabla^{(m)}R_{ijkl}$ which have a factor $\nabla\phi_h$ contracting against an internal index. For any such factor F_h, we denote by $\Pi_l(h)$ (or $\Pi_j(h)$) the set of numbers π for which $\nabla\phi_\pi$ is contracting against an internal index in the factor F_h. We will denote by Π (or Π_l,Π_f if we wish to be more precise) the set of all numbers π for which $\nabla\phi_\pi$ is contracting against an internal index in some curvature factor $\nabla^{(m)}R_{ijkl}$. For the purposes of the next definition we assume that if $\pi\in\Pi$ then the factor $\nabla\phi_\pi$ is contracting against the index $_i$ of a factor $\nabla^{(m)}R_{ijkl}$. This assumption can be made with no loss of generality for the purposes of this proof. If it were contracting against one of the indices $_j,{}_k,{}_l$, we can apply standard identities to make it contract against the index $_i$.

Now, for each $l\in L$ and each $\pi\in\Pi_l(h)$ and each given complete contraction $C_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ in $Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$, we define an operation $Oper^{\pi,k,h}_{\phi_{u+1}}[C_g]$ that acts as follows: If the index $_k$ in F_h is contracting against the selected factor, $Oper^{\pi,k,h}_{\phi_{u+1}}[C_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ stands for the complete contraction that arises from $C_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ by replacing the expression $\nabla^{(m)}_{r_1\dots r_m}R_{ijkl}\nabla^i\phi_\pi$ by $-\nabla_k\phi_{u+1}\nabla^{(m)}_{r_1\dots r_m}R_{ijal}\nabla^i\phi_\pi\nabla^a\tau$. If $_k$ is not contracting against the selected factor, we let $Oper^{\pi,k,h}_{\phi_{u+1}}[C_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=0$. Similarly, we define the operation $Oper^{\pi,l,h}_{\phi_{u+1}}[\dots]$.

Then, for each $j\in J$ and each $\pi\in\Pi_j(h)$, we define an operation $Oper^{\pi,k,h}_{\phi_{u+1}}[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ that acts as follows: If $_k$ (in F_h) is contracting against the selected factor, we define $Oper^{\pi,k,h}_{\phi_{u+1}}[C^f_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$,

to stand for the complete contraction that arises from $C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)$ by replacing the expression $\nabla^{(m)}_{r_1 \dots r_m} R_{ijkl} \nabla^i \phi_f$ by an expression $-\nabla_k \phi_{u+1} \nabla^{(m)}_{r_1 \dots \dots r_m} R_{ijal} \nabla^i \phi_\pi \nabla^a \tau$. If $_k$ is not contracting against the selected factor, we define $Oper^{\pi,k,h}_{\phi_{u+1}}[C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)] = 0$. Similarly, we define the operation $Oper^{\pi,l,h}_{\phi_{u+1}}[C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)]$. This operation extends to linear combinations.

We then define, for each $l \in L$:

$$\begin{aligned} & Special_{\phi_{u+1}}[Xdiv_{i_1} \dots Xdiv_{i_a} C^{l,i_1 \dots i_a}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)] = \\ & \sum_{\pi \in \Pi(l)} G^{\tau,\pi} \{ Oper^{\pi,k,h}_{\phi_{u+1}}[Xdiv_{i_1} \dots Xdiv_{i_a} C^{l,i_1 \dots i_a}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)] + \\ & Oper^{\pi,l,h}_{\phi_{u+1}}[Xdiv_{i_1} \dots Xdiv_{i_a} C^{l,i_1 \dots i_a}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)] \}, \end{aligned} \tag{2.31}$$

and also for each $j \in J$:

$$\begin{aligned} & Special_{\phi_{u+1}}[C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)] = \\ & \sum_{\pi \in \Pi(f)} G^{\tau,\pi} \{ Oper^{\pi,k,h}_{\phi_{u+1}}[C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)] + \\ & Oper^{\pi,l,h}_{\phi_{u+1}}[C^f_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)] \}. \end{aligned} \tag{2.32}$$

Thus, by construction each tensor field or complete contraction in $Special_{\phi_{u+1}}[Xdiv_{i_1} \dots Xdiv_{i_a} C^{l,i_1 \dots i_a}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)]$ or $Special_{\phi_{u+1}}[C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)]$ has length $\sigma+u$ and an internal contraction in a factor $\nabla^{(p)} Ric_{ik}$. Moreover, we observe that each complete contraction in $Special_{\phi_{u+1}}[C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)]$ will have at least one factor $\nabla \phi_b$, $b \in Def(\vec{\kappa}_{simp})$[47] contracting against a derivative index of a factor $\nabla^{(m)} R_{ijkl}$ or $\nabla^{(p)} Ric$.

We then claim:

[47]Recall that $Def(\vec{\kappa}_{simp})$ stands for the the set of numbers h for which some factor $\nabla \tilde{\phi}_h$ is contracting against the index $_i$ in some $S_* \nabla^{(\nu)} R_{ijkl}$.

Lemma 2.3 *In the notation above, we claim that for each $h, 1 \le h \le X$:*

$$
\begin{aligned}
&(0=)\sum_{t\in T^h} a_t \sum_{\rho\in Set(h)} G^{\rho,\tau}\{C^t_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\tau)\}+\\
&\sum_{y\in Y^h} a_y \sum_{\rho\in Set(h)} G^{\rho,\tau}\{C^y_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\tau)\} =\\
&\sum_{l\in L} a_l G^\sharp\{LC_\Phi[Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}\\
&+Special_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}+\\
&\sum_{j\in J} a_j G^\sharp\{LC_\Phi[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}\\
&+Special_{\phi_{u+1}}[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)].
\end{aligned}
\tag{2.33}
$$

Thus, in view of equations (2.24), (2.25) the above Lemma in some sense provides us with information on the linear combination

$$
\begin{aligned}
&\sum_{l\in L} a_l LC_\Phi[Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&\sum_{j\in J} a_j LC_\Phi[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)].
\end{aligned}
\tag{2.34}
$$

Proof of Lemma 2.3: We see our claim by the definition of the operation $Difference[\dots]$, by book-keeping and also from the definitions of the operations G and $G^\sharp$.

(Brief discussion:) Notice that for any complete contraction C_g in $Xdiv_{i_1},\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g$ or C^j_g, the complete contractions appearing in $Difference[C_g]$ arise exclusively by applying the third term in the transformation law (2.2) to two indices $(\nabla_{s},{}_z)$[48] in some factor F_k in C_g (provided the factor $\nabla\phi_{u+1}$ that we then introduce contracts against the selected factor). We then obtain a complete contraction $C'_g(\nabla\phi_{u+1})$ that arises from C_g by replacing the expression $(\nabla_s,{}_z)\nabla^s\tau$ by an expression $-\nabla_z\phi_{u+1}({}_s,\nabla^s\tau)$ (here the factor $\nabla^s\tau$ is now contracting against the "position" that the index ${}_z$ occupied). Furthermore, consider any factor $\nabla\phi_\rho$ that is contracting against the factor F_k and denote by ${}_b$ the index against which it contracts.

Now consider the case where one of the indices ${}_z, {}_b$ described in the previous paragraph are derivative indices. We then observe that if we replace two factors $\nabla_\alpha\phi_\rho, \nabla_\beta\tau$ in $C'_g(\nabla\phi_{u+1})$ by $g_{\alpha\beta}$, we obtain precisely the complete contraction that arises in $LC_\Phi[C_g]$ when we apply the second or third term of the transformation law (2.2) to the indices ${}_z, {}_b$ and then replace the factor $\nabla\phi_\rho$ by an internal contraction (as described in the definition of the operation $G^\sharp$). It is easy to see that this gives a one-to-one correspondence between the LHS of

[48](where ∇_s is contracting against the factor $\nabla^s\tau$ that we have introduced–i.e. provided that ${}_z$ contracts against the selected factor in C_g)

(2.33) and the terms $G^\sharp[\dots]$ in the RHS of (2.33), *except* for the terms arising in the LHS when both the indices $_b, _z$ discussed above are internal indices. The same "book-keeping" reasoning then shows that the terms we obtain from the LHS of (2.33) correspond to the terms $Special_{\phi_{u+1}}[\dots]$ in the RHS of (2.33). $\Box$

The goal of our next Lemma will be to "get rid" of the sublinear combination

$$\begin{aligned}&\sum_{l\in L} a_l Special_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\\&+\sum_{j\in J} a_j Special_{\phi_{u+1}}[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\end{aligned} \tag{2.35}$$

in (2.33), and to replace it by a new linear combination which will have one internal contraction involving a derivative index (rather than an internal contraction in a factor $\nabla^{(p)}Ric$). To state our next Lemma, we will be using tensor fields and complete contractions with an internal contraction in a factor $\nabla^{(m)}R_{ijkl}$, where that internal contraction will involve a derivative index. It will be useful to recall the operation Sub_ω from the Appendix in [3] and the discussion directly above it.

In particular, we will show the following:

Lemma 2.4 *In the notation of equation (2.33), we claim that we can write:*

$$\begin{aligned}&\sum_{l\in L} a_l Special_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\\&+\sum_{j\in J} a_j Special_{\phi_{u+1}}[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] =\\&\sum_{t\in T} a_t Xdiv_{i_1}\dots Xdiv_{i_a} C^{t,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})\\&+\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),\end{aligned} \tag{2.36}$$

where the complete contractions and tensor fields on the right hand side have an internal contraction in exactly one factor $\nabla^{(m)}R_{ijkl}$ and that internal contraction involves a derivative index. Furthermore, in each tensor field and each complete contraction in the above equation precisely one of the factors $\nabla\phi_x, x = 1\dots,u$ is missing. We accordingly denote by T^x, J^x the corresponding index sets of the complete contractions and tensor fields where $\nabla\phi_x$ is missing.

We furthermore claim that in (2.36) each of the tensor fields $Sub_{\phi_x}\{C^{t,i_1\dots i_a}_g\}$ has the following properties: In the case of Lemma 1.1 and 1.2 it is acceptable and moreover has a $(u+1)$-simple character $\vec{\kappa}^+_{simp}$. In the case of Lemma 1.3,

$$\sum_{t\in T^x} a_t Sub_{\phi_x}\{Xdiv_{i_1}\dots Xdiv_{i_a} C^{t,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})\}$$

will be a generic linear combination of complete contractions like the ones indexed in $T_1 \bigcup T_2$ *(in the conclusion of that Lemma).*

On the other hand, for each $j \in J^x$, $Sub_{\phi_x}[C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_{u+1})]$ *is simply subsequent to* $\vec{\kappa}^+_{simp}$.

Let us show how proving Lemma 2.4, in conjunction with equation (2.33) (which we have already proven), would imply our Lemma 2.2.

Proof that Lemma 2.4 implies Lemma 2.2:

By virtue of the Lemma 2.4 we may refer to (2.33) and replace:

$$\sum_{l \in L} a_l Special_{\phi_{u+1}}[Xdiv_{i_1} \dots Xdiv_{i_a} C^{l,i_1 \dots i_a}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)]$$

as in the conclusion of Lemma 2.4.

We then obtain an equation:

$$\begin{aligned}
0 = &\sum_{l \in L} a_l G^\sharp\{LC_\Phi[Xdiv_{i_1} \dots Xdiv_{i_a} C^{l,i_1 \dots i_a}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)]\}+ \\
&\sum_{j \in J} a_j G^\sharp\{LC_\Phi[C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)]\}+ \\
&\sum_{t \in T} a_t Xdiv_{i_1} \dots Xdiv_{i_a} C^{t,i_1 \dots i_a}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_{u+1}) \\
&+ \sum_{j \in J} a_j C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_{u+1}),
\end{aligned} \tag{2.37}$$

where the linear combinations indexed in T, J are as in (2.36).

We can then derive Lemma 2.2 straightforwardly: We break the above equation into sublinear combinations according to the factor $\nabla\phi_x$ that is missing. Each of these sublinear combinations must vanish separately, since the above holds formally. We denote the respective sublinar combinations in each $G^\sharp\{\dots\}$ by $G^{\sharp,x}\{\dots\}$. Thus we have that for each x:

$$\begin{aligned}
0 = &\sum_{l \in L} a_l G^{\sharp,x}\{LC_\Phi[Xdiv_{i_1} \dots Xdiv_{i_a} C^{l,i_1 \dots i_a}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)]\}+ \\
&\sum_{j \in J} a_j G^{\sharp,x}\{LC_\Phi[C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)]\}+ \\
&\sum_{t \in T^x} a_t Xdiv_{i_1} \dots Xdiv_{i_a} C^{t,i_1 \dots i_a}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_{u+1}) \\
&+ \sum_{j \in J^x} a_j C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_{u+1}).
\end{aligned} \tag{2.38}$$

We then use the operation Sub_{ϕ_x} (which acts on each complete contraction C_g in the right hand side of (2.37) by picking out the one factor $\nabla^{r_a}\nabla^{(m)}_{r_1\dots r_m}R_{r_{m+1}\dots r_{m+4}}$ or $\nabla^{r_a}\nabla^{(p)}_{r_1\dots r_p}\Omega_h$ with the internal contraction,[49] and replacing it by an expression $\nabla^{(m)}_{r_1\dots r_m}R_{r_{m+1}\dots r_{m+4}}\nabla^{r_a}\phi_x$, $\nabla^{(p)}_{r_1\dots r_p}\Omega_h\nabla^{r_a}\phi_x$).

Now, by just keeping track of the definitions above we find that:

$$\sum_{x=1}^{u} Sub_{\phi_x}\{G^{\sharp,x}\{LC_\Phi[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}\} = LC_\Phi[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)], \tag{2.39}$$

and also, for every $j\in J$:

$$\sum_{x=1}^{u} Sub_{\phi_x}\{G^{\sharp,x}\{LC_\Phi[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\} = LC_\Phi[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]. \tag{2.40}$$

Hence, acting by $\sum_x Sub_{\phi_x}$ on (2.37), we deduce our Lemma 2.2. $\Box$

Therefore, if we can show Lemma 2.4, we can then derive Lemma 2.2.

Proof of Lemma 2.4:

Firstly we make a few observations: The complete contractions and tensor fields in the sublinear combinations $Special_{\phi_{u+1}}[\dots]$ are each in the form:

$$pcontr(\nabla^{(m_1)}R_{ijkl}\otimes\dots\otimes\nabla^{(m_{\sigma_1})}R_{ijkl}\otimes\nabla^{(b)}Ric_{ij}\otimes S_*\nabla^{(\nu_1)}R_{ijkl}\otimes \dots\otimes S_*\nabla^{(\nu_t)}R_{ijkl}\otimes\nabla^{(b_1)}\Omega_1\otimes\dots\otimes\nabla^{(b_p)}\Omega_p\otimes \nabla\phi_{z_1}\dots\otimes\nabla\phi_{z_u}\otimes\nabla\phi'_{z_{u+1}}\otimes\dots\otimes\nabla\phi'_{z_{u+d}}\otimes\dots\otimes\nabla\tilde{\phi}_{z_{u+d+1}}\otimes\dots\otimes\nabla\tilde{\phi}_{z_{u+d+y}}), \tag{2.41}$$

where for each of the complete contractions and tensor fields in the above form one of the factors $\nabla\phi_1,\dots,\nabla\phi_u$ is missing (the factor $\nabla\phi_x$ in the notation of Lemma 2.4). Accordingly, we will re-express the LHS of our Lemma hypothesis as:

[49] The indices $(\nabla^{r_a}, {}_{r_a})$ are contracting against each other.

$$\sum_{l\in L} a_l Special_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$
$$+\sum_{j\in J} a_j Special_{\phi_{u+1}}[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=$$
$$\sum_{x=1}^{X}\{\sum_{l\in L'_x} a_l Xdiv_{i_1}\dots Xdiv_{i_c} C^{l,i_1\dots i_c,i_b}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_b}\phi_{u+1}+$$
$$\sum_{j\in J'_x} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})\}, \tag{2.42}$$

where complete contractions and tensor fields indexed in L'_x, J'_x have the factor $\nabla\phi_x$ missing. (We will explain momentarily how c is related to a).

We will then prove the assertion of Lemma 2.4 for each of the sublinear combinations indexed in L'_x, J'_x separately. Clearly, just adding all those equations will then show our whole claim.

Recall (from the Appendix in [3]) the operation $Ricto\Omega$ (also denoted by $UnRic$) which replaces the factor $\nabla^{(p)}_{r_1\dots r_p}Ric_{ik}$ by a factor $\nabla^{(p+2)}_{r_1\dots r_p ik}\Omega_{p+1}$. In view of (2.33) we derive that for every $x\le u$:

$$\sum_{l\in L'_x} a_l Xdiv_{i_1}\dots Xdiv_{i_c} C^{l,i_1\dots i_c,i_b}_g(\Omega_1,\dots,\Omega_{p+1},\phi_1,\dots,\phi_u)\nabla_{i_b}\phi_{u+1}+$$
$$\sum_{j\in J'_x} a_j C^j_g(\Omega_1,\dots,\Omega_{p+1},\phi_1,\dots,\phi_{u+1})=0, \tag{2.43}$$

modulo complete contractions of length $\ge \sigma+u+2$. This holds because (2.42) holds formally.

We will be using the equations (2.43) to derive our claim. We have two different proofs based on x: Either $x\in Def(\vec{\kappa}_{simp})$ or $x\notin Def(\vec{\kappa}_{simp})$.[50] We start with the case where $x\in Def(\vec{\kappa}_{simp})$.

Proof of Lemma 2.4 in the case $x\in Def(\vec{\kappa}_{simp})$:

In this case we will derive our claim in two steps. Firstly, (in the notation of (2.42)) we claim that we can write:

[50] Recall that $Def(\vec{\kappa}_{simp})$ stands for the set of numbers o for which some $\nabla\tilde{\phi}_o$ is contracting against the index $_i$ in some factor $S_*\nabla^{(\nu)}R_{ijkl}$.

$$\sum_{l\in L'_x} a_l Xdiv_{i_1}\dots Xdiv_{i_c} C_g^{l,i_1\dots i_c,i_b}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_b}\phi_{u+1} =$$
$$\sum_{t\in T} a_t Xdiv_{i_1}\dots Xdiv_{i_a} C_g^{t,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})+$$
$$\sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}) + \sum_{j\in J'} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}), \quad (2.44)$$

where the tensor fields and complete contractions indexed in T, J are as in the conclusion of Lemma 2.4, while the contractions indexed in J' have exactly one factor $\nabla^{(p)}Ric$ but also have one of the factors $\nabla\phi_y, y \in Def(\vec{\kappa}_{simp})$[51] contracting against a derivative index in a factor $\nabla^{(m)}R_{ijkl}$ *or, in the case of Lemmas* 1.1, 1.2, the factor $\nabla\phi_{u+1}$ is contracting against a *derivative index* of the (one of the) selected factor(s) $\nabla^{(m)}R_{ijkl}$.

If we can show (2.44), then by replacing the above into (2.33) and using the notation of (2.42), we obtain a new equation:

$$\sum_{l\in L} a_l G^{\sharp,x}\{LC_\Phi[Xdiv_{i_1}\dots Xdiv_{i_a} C_g^{l,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}+$$
$$\sum_{j\in J} a_j G^{\sharp,x}\{LC_\Phi[C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}+$$
$$\sum_{t\in T} a_t Xdiv_{i_1}\dots Xdiv_{i_a} C_g^{t,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})+$$
$$\sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})+$$
$$\sum_{j\in J'} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}) + \sum_{j\in J'_x} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}) = 0; \quad (2.45)$$

here the linear combination indexed in J' is not generic notation, it is precisely the linear combination appearing in the right hand side of (2.44). Furthermore, we observe that the complete contractions belonging to the sublinear combination

$$\sum_{j\in J'_x} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$$

will have at least one of the factors $\nabla\phi_x, x \in Def(\vec{\kappa}_{simp})$ contracting against a derivative index of some factor $\nabla^{(m)}R_{ijkl}$. This follows by the definition of the operation $Special_{\phi_{u+1}}[\dots]$.

Now, applying the operation $Ricto\Omega_{p+1}$ to the above we derive an equation:

[51] Recall that $Def(\vec{\kappa}_{simp})$ stands for the set of numbers ρ for which some factor $\nabla\phi_\rho$ is contracting against the index $_i$ in a factor $S_*\nabla^{(\nu)}R_{ijkl}$.

$$\sum_{j\in J'} a_j C^j_g(\Omega_1,\dots,\Omega_{p+1},\phi_1,\dots,\phi_{u+1}) + \sum_{j\in J'_x} a_j C^j_g(\Omega_1,\dots,\Omega_{p+1},\phi_1,\dots,\phi_{u+1}) = 0, \tag{2.46}$$

where $C^j(\Omega_1,\dots,\Omega_{p+1},\phi_1,\dots,\phi_{u+1})$ arises from $C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$ by replacing the factor $\nabla^{(y)}Ric$ by $\nabla^{(y+2)}\Omega_{p+1}$.

Since the above holds formally, we may repeat the permutations by which we make it vanish formally to the linear combination

$$\sum_{j\in J'} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}) + \sum_{j\in J'_x} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots \\ ,\phi_{u+1}) = 0, \tag{2.47}$$

modulo introducing correction terms of length $\sigma+u+1$ by virtue of the formula $\nabla_a Ric_{bc} - \nabla_b Ric_{ac} = \nabla^l R_{ablc}$, and also correction terms of length $\sigma+u+2$ (which we do not care about). Thus, we derive that we can write:

$$\sum_{j\in J'} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}) + \sum_{j\in J'_x} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}) \\ = \sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}). \tag{2.48}$$

Therefore, in the case $x\in Def(\vec{\kappa}_{simp})$, matters are reduced to showing (2.44).

Proof of (2.44):

Now, we see that since all $C^{l,i_1,\dots,i_a}_g$, $l\in L$ in (2.33) have a given simple character $\vec{\kappa}_{simp}$, it follows that all tensor fields $C^{l,i_1\dots i_c,i_b}(\Omega_1,\dots,\Omega_{p+1},\phi_1,\dots,\phi_u)$, $l\in L'_x$ in (2.44) will have the same $(u-1)$-simple character (the one defined by the factors $\nabla\phi_1,\dots,\hat{\nabla}\phi_x,\dots,\nabla\phi_u$).[52] We denote this $(u-1)$-simple character by $\vec{\kappa}^x_{-1}$. Moreover, each $C^j_g(\Omega_1,\dots,\Omega_{p+1},\phi_1,\dots,\phi_u)$, $j\in J'_x$ must be $(u-1)$-subsequent to $\vec{\kappa}^x_{-1}$.

Now, an observation: In the language of the introduction in [6], if (1.6) falls under case II or III (i.e. if we are proving Lemma 1.2 or Lemma 1.3) then for each $l\in L_\mu$ we have $Def^*(l)=0$ (i.e. in the tensor fields $C^{l,i_1\dots i_\mu}_g$ in (1.6) have no special free indices in any factor $S_*\nabla^{(\nu)}R_{ijkl}$). Thus, we see by construction that in (2.42) each c is $\geq\mu$. On the other hand, in the setting of Lemma 1.1, we have that for any F_h in the form $S_*\nabla^{(\nu)}R_{ijkl}$ for which $_k$ is a free index then we may write out:

[52] See the notation in (2.42).

$$G^{\tau,\pi}\{Oper^{k,h}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}\}=\sum_{r\in R}a_rXdiv_{i_1}\dots Xdiv_{i_{a-1}}C^{r,i_1\dots i_{a-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}). \tag{2.49}$$

Note that if $l\in L_\mu$ (i.e. if $a=\mu$) then the tensor fields on the right hand side will have the factor $\nabla\phi_{u+1}$ contracting against a derivative index in the selected factor. Thus, in the notation we introduced:

$$\sum_{r\in R}a_rXdiv_{i_1}\dots Xdiv_{i_{a-1}}C^{r,i_1\dots i_{a-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}) = \sum_{j\in J'}a_jC^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}). \tag{2.50}$$

Therefore, after this observation, we may assume that all the tensor fields indexed in each L'_x in the equation (2.42) have $c\ge\mu$. Now, we will prove our Lemma by an induction: We will use the notation

$$\sum_{t\in T}a_tXdiv_{i_1}\dots Xdiv_{i_a}C^{t,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),$$

$$\sum_{j\in J\bigcup J'}a_jC^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$$

to denote generic linear combinations as described above. We assume that for some $m\ge\mu$:

$$\begin{aligned}&\sum_{l\in L'_x}a_lXdiv_{i_1}\dots Xdiv_{i_c}C^{l,i_1\dots i_c,i_b}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_b}\phi_{u+1}=\\&\sum_{d\in D^m}c_dXdiv_{i_1}\dots Xdiv_{i_{a_d}}C^{d,i_1\dots i_{a_d}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})+\\&\sum_{t\in T}a_tXdiv_{i_1}\dots Xdiv_{i_a}C^{t,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})\\&+\sum_{j\in J\bigcup J'}a_jC^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),\end{aligned} \tag{2.51}$$

where here the vector fields $C^{d,i_1\dots i_{a_d}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})$ are in the form (2.41) with a factor $\nabla^{(p)}Ric$ and each has $a_d\ge m$. Moreover, they each have a $(u-1)$-simple character $\vec{\kappa}^x_{-1}$. We then claim that we can write:

$$\sum_{l\in L'_x} a_l Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a,i_b}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_b}\phi_{u+1} =$$
$$\sum_{d\in D^{m+1}} a_d Xdiv_{i_1}\dots Xdiv_{i_{a_d}} C^{d,i_1\dots i_{a_d}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})+$$
$$\sum_{t\in T} a_t Xdiv_{i_1}\dots Xdiv_{i_a} C^{t,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}) \tag{2.52}$$
$$+\sum_{j\in J\bigcup J'} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),$$

with the same notational conventions as above.

Clearly, if we can show the above inductive statement then by iterative repetition we will derive (2.44).

Proof that (2.51) implies (2.52): We observe that all the tensor fields in (2.51) have the same $(u-1)$-simple character, which we have denoted by $\vec{\kappa}^x_{-1}$.

Now, by applying $Ricto\Omega_{p+1}$ (see the Appendix in [3]) to (2.51) and using (2.43) we derive:

$$\sum_{d\in D^m} c_d Xdiv_{i_1}\dots Xdiv_{i_{a_d}} Ricto\Omega_{p+1}[C^{d,i_1\dots i_{a_d}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})]+$$
$$\sum_{j\in J'_x} a_j C^j_g(\Omega_1,\dots,\Omega_{p+1},\phi_1,\dots,\phi_u,\phi_{u+1})$$
$$+\sum_{j\in J'} a_j Ricto\Omega_{p+1}[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=0, \tag{2.53}$$

modulo complete contractions of length $\geq \sigma+u+1$.

We pick out the index set $D^{m,*}\subset D^m$ of those tensor fields for which $a_d=m$. Then (apart from certain "forbidden cases" which we discuss in the "Digression" below), we apply the first claim of Lemma 4.10 in [3] to (2.53),[53] we deduce that for some linear combination of $(m+1)$-tensor fields with $(u-1)$-simple character $\vec{\kappa}^x_{-1}$ (say $\sum_{s\in S} a_s C^{s,i_1\dots i_{m+1}}_g(\Omega_1,\dots,\Omega_{p+1},\phi_1,\dots,\phi_u,\phi_{u+1})$), we will have that:

[53]Note that (2.53) formally falls under the inductive assumption of this Lemma, because we replaced a curvature term by a factor $\nabla^{(y)}\Omega_{p+1}$, hence we are in a case in which case Corollary 1 in [6] already holds, by our inductive assumption. Observe that if $m=\mu$ there is no danger of falling under a "forbidden case" by weight considerations, since we are assuming that the equation in our Lemma assumption does not contain "forbidden terms". The possibility of "forbidden cases" when $m>\mu$ will be treated in the "Digression" below.

$$\sum_{d\in D^{m,*}} a_d Ricto\Omega_{p+1}[C^{d,i_1\dots i_m}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})]\nabla_{i_1}\upsilon\dots\nabla_{i_m}\upsilon$$
$$- Xdiv_{i_{m+1}}\sum_{s\in S} a_s C^{s,i_1\dots i_{m+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})\nabla_{i_1}\upsilon\dots\nabla_{i_m}\upsilon$$
$$= \sum_{j\in J'} a_j Ricto\Omega_{p+1}[C^{j,i_1\dots i_m}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_m}\upsilon];
\tag{2.54}$$

(we are using the same generic notational conventions as above-the m-tensor fields $Ricto\Omega_{p+1}[C^{j,i_1\dots i_m}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ again have a $(u-1)$-simple character that is simply subsequent to $\vec{\kappa}^x_{-1}$). Therefore, since the above must hold formally, we conclude that:

$$\sum_{d\in D^{m,*}} a_d Xdiv_{i_1}\dots Xdiv_{i_m} C^{d,i_1\dots i_m}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})-$$
$$Xdiv_{i_1}\dots Xdiv_{i_m}Xdiv_{i_{m+1}}\sum_{s\in S} a_s C^{s,i_1\dots i_{m+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1}) =$$
$$\sum_{t\in T} a_t Xdiv_{i_1}\dots Xdiv_{i_a} C^{t,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})+$$
$$\sum_{j\in J\bigcup J'} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),
\tag{2.55}$$

where $C^{s,i_1\dots i_{m+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})$ arises from $C^{s,i_1\dots i_{m+1}}_g(\Omega_1,\dots,\Omega_{p+1},\phi_1,\dots,\phi_u,\phi_{u+1})$ by formally replacing the factor $\nabla^{(p)}_{r_1\dots r_p}\Omega_{p+1}$ by a factor $\nabla^{(p-2)}_{r_1\dots r_{p-2}}Ric_{r_{p-1}r_p}$. Furthermore, $\sum_{t\in T}\dots$ is a generic linear combination as described after (2.44).

This is precisely our desired inductive step. Therefore we have derived our claim in the case where $x\in Def(\vec{\kappa}_{simp})$, except for the "forbidden cases" which we now discuss:

Digression: The "forbidden cases". As noted above, the only case where Lemma 4.10 in [6] cannot be applied to (2.53) (because it falls under a "forbidden case" of that Lemma) is when $m>\mu$. So, in that case we derive from (2.51) that $D^{m,*}=D^m$, and then applying the "weak substitute" of the fundamental Proposition 2.1 in [6] we derive:[54]

[54]See the Appendix of [6].

$$\sum_{d\in D^{m,*}} a_d Xdiv_{i_1}\dots Xdiv_{i_m} C^{d,i_1\dots i_m}_g(\dots,\phi_{u+1}) =$$
$$\sum_{t\in T} a_t Xdiv_{i_1}\dots Xdiv_{i_{m-1}} C^{t,i_1\dots i_{m-1}}_g(\dots,\phi_{u+1})+$$
$$\sum_{j\in J} a_j C^j_g(\dots,\phi_{u+1}) + \sum_{f\in F} a_f Xdiv_{i_1}\dots Xdiv_{i_{m-1}} C^{f,i_1\dots i_{m-1}}_g(\dots,\phi_{u+1}), \tag{2.56}$$

where the terms indexed in J are simply subsequent to $\vec{\kappa}_{simp}$ and have a factor Ric, while the terms in F have the $\nabla\phi_{u+1}$ contracting against a non-special index (and both terms above have a factor Ric).

Then, replacing the above into (2.51) and applying Lemma 4.10 in [6] (notice it can now be applied, since the factor $\nabla\phi_{u+1}$ is contracting against a non-special index), we derive that:

$$\sum_{f\in F} a_f C^{f,i_1\dots i_{m-1}}_g(\dots,\Omega_{p+1},\phi_{u+1})\nabla_{i_1}\upsilon\dots\nabla_{i_{m-1}}\upsilon = 0.$$

(Here $C^{f,i_1\dots i_{m-1}}_g(\dots,\Omega_{p+1},\phi_{u+1})$ arises from $C^{f,i_1\dots i_{m-1}}_g(\dots,\phi_{u+1})$ by applying $Ricto\Omega_{p+1}$).

Thus, we may erase the terms $\sum_{f\in F}\dots$ in (2.56); with that new feature, (2.56) is precisely our desired equation (2.55). □

Proof of Lemma 2.4 in the case $x\notin Def(\vec{\kappa}_{simp})$:

We recall that the factor $\nabla\phi_x$ ($x\notin Def(\vec{\kappa}_{simp})$) is contracting against a factor $T^*(x)=\nabla^{(m)}R_{ijkl}$. We then distinguish two cases: Either in $\vec{\kappa}_{simp}$ there is some other $h'\neq x$ with $h'\in Def(\vec{\kappa}_{simp})$ so that $\nabla\phi_{h'}$ is contracting against $T^*(x)$ in $\vec{\kappa}_{simp}$, or there is no such factor. Another way of describing these two cases is that in the first case the factor $T^*(x)=\nabla^{(m)}R_{ijkl}$ has arisen from the de-symmetrization of some factor $S_*\nabla^{(\nu)}R_{ijkl}$ (for which the factor $\nabla\phi_{h'}$ was contracting against the index $_i$), while in the second case $\nabla^{(m)}R_{ijkl}$ corresponds to a factor $\nabla^{(m)}R_{ijkl}$ in $\vec{\kappa}_{simp}$. The second subcase is easier, so we will start with that one.

Second subcase: We observe that in this setting we must have $L'_x=\emptyset$ (refer to (2.43)). Moreover, for each C^j_g, $j\in J'_h$, we must have at least one of the factors $\nabla\phi_w$, $w\in Def(\vec{\kappa}_{simp})$ contracting against a derivative index of some factor F_b in $C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\hat{\phi}_x,\dots,\phi_u,\phi_{u+1})$, where in addition $F_b\neq T^*(x)$. In view of these observations, it is enough to show that in this second subcase:

$$\sum_{j\in J'_x} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\hat{\phi}_x,\dots,\phi_u,\phi_{u+1}) \\ = \sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\hat{\phi}_x,\dots,\phi_u,\phi_{u+1}), \tag{2.57}$$

modulo complete contractions of greater length. Clearly, that will imply our claim for this second subcase since $L'_x = \emptyset$ in (2.43). We derive this equation rather easily: By (2.43) we have:

$$\sum_{j\in J'_x} a_j C^j_g(\Omega_1,\dots,\Omega_{p+1},\phi_1,\dots,\hat{\phi}_h,\dots,\phi_u,\phi_{u+1}) = 0, \tag{2.58}$$

where we recall that for each C^j_g, $j\in J'_x$, one of the factors $\nabla\phi_c$, $c\in Def(\vec{\kappa}_{simp})$ is contracting against a derivative index of some curvature factor.

Now, since (2.58) holds formally (at the linearized level), we may repeat the permutations by which we make the left hand side vanish formally to the linear combination

$$\sum_{j\in J'_x} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\hat{\phi}_h,\dots,\phi_u,\phi_{u+1}),$$

and derive the right hand side in (2.57) as correction terms.

First subcase: Recall that we are assuming $x\notin Def(\vec{\kappa}_{simp})$, and moreover the factor $\nabla\phi_x$ is contracting against some factor $T^*(x)$ in $\vec{\kappa}_{simp}$ in the form $S_*\nabla^{(\nu)}R_{ijkl}$. Recall also that we are now assuming that $\nabla\phi_x$ is *not* the factor that contracts against the index $_i$ in $T^*(x)$ in $\vec{\kappa}_{simp}$. That factor is $\nabla\phi_{h'}$.

In this first subcase we define $\sum_{j\in J''} a_j\dots$ to stand for a generic linear combination of complete contractions in the form (2.41) with the factor $\nabla\phi_{h'}$ contracting against a *derivative* index in a factor $\nabla^{(p)}Ric$. We also denote by

$$\sum_{x\in X} a_x C^{x,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\hat{\phi}_h,\dots,\phi_u,\phi_{u+1})$$

a generic linear combination of tensor fields in the form (2.41) with a factor Ric_{ij} (with no derivatives) where $\nabla\phi_{h'}$ is contracting against the index $_i$ in that factor Ric_{ij}, and where $a\geq\mu$. Finally, we define $\sum_{j\in J'} a_j\dots$ to stand for the same generic linear combination as in the previous case.

It then follows that in this first subcase we can re-express the terms in the equation (2.42) as:

$$\sum_{l\in L'_x} a_l Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a,i_b}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_b}\phi_{u+1}+$$
$$\sum_{j\in J'_x} a_j C^j_g(\Omega_1,\dots,\Omega_{p+1},\phi_1,\dots,\phi_{u+1}) =$$
$$\sum_{t\in T} a_t Xdiv_{i_1}\dots Xdiv_{i_a} C^{t,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})+$$
$$Xdiv_{i_1}\dots Xdiv_{i_a}\sum_{x\in X} a_x C^{x,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\hat{\phi}_h,\dots,\phi_u,\phi_{u+1})+$$
$$\sum_{j\in J\bigcup J'\bigcup J''} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}). \tag{2.59}$$

This just follows by the definitions and by applying the second Bianchi identity to the factor $\nabla^{(p)}Ric$ if necessary (this can be done, because $p>0$).

We then prove Lemma 2.4 in this setting via an inductive statement: Let us suppose that the minimum rank among the tensor fields indexed in X in (2.59) is $m\ge\mu$ and the corresponding tensor fields are indexed in $X^m\subset X$. We then claim that we can write:

$$Xdiv_{i_1}\dots Xdiv_{i_m}\sum_{x\in X^m} a_x C^{x,i_1\dots i_m}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\hat{\phi}_h,\dots,\phi_u,\phi_{u+1}) =$$
$$Xdiv_{i_1}\dots Xdiv_{i_{m+1}}\sum_{x\in X^{m+1}} a_x C^{x,i_1\dots i_{m+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\hat{\phi}_h,\dots,\phi_u,\phi_{u+1})+$$
$$\sum_{t\in T} a_t Xdiv_{i_1}\dots Xdiv_{i_a} C^{t,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})+$$
$$\sum_{j\in J\bigcup J'\bigcup J''} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}), \tag{2.60}$$

where for all the linear combinations on the right hand side we are using generic notation.

We will show (2.60) momentarily. For the time being, we observe that if we can show (2.60) then by iterative repetition we are reduced to proving our claim for the first subcase above under the extra assumption that $X=\emptyset$.

Under this extra assumption we claim that the sum in $J'\bigcup J''$ in the above satisfies:

$$\sum_{j\in J'\bigcup J''} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}) = \sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}). \tag{2.61}$$

Notice that proving the above two equations will complete the proof of Lemma 2.4 in this subcase. We first prove (2.61) (assuming we have shown (2.60)) by the usual argument:

Plug (2.60) into (2.59) and then apply $Ricto\Omega_{p+1}$ to derive:

$$\sum_{j\in J'\bigcup J''} a_j C^j_g(\Omega_1,\dots,\Omega_{p+1},\phi_1,\dots,\phi_{u+1}) = 0.$$

Now, we use the fact that the resulting equation holds formally: We may arrange that the factor $\nabla\phi_{h'}$ is contracting against the first index in the factor $\nabla^{(p)}_{r_1\dots r_p}Ric_{ij}$, hence in the permutations by which we make the LHS of the above formally zero the first index is not moved. Thus we see that the correction terms arising when we repeat those permutations for

$$\sum_{j\in J'\bigcup J''} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$$

are indeed as in the right hand side of (2.61).

Proof of (2.60): First of all, observe that if the factor $S_*R_{ijkl}\nabla^i\tilde{\phi}_h$ in $\vec{\kappa}_{simp}$ is contracting against some factor $\nabla\phi_{h''}$ (in addition to the factors $\nabla\phi_h,\nabla\phi_{h'}$), then (2.60) is obvious since then by definition $X^m = X = \emptyset$. Thus, we may now assume that only the factors $\nabla\phi_x$ and $\nabla\phi_{h'}$ are contracting against $S_*R_{ijkl}\nabla^i\tilde{\phi}_h$ in $\vec{\kappa}_{simp}$.

In that setting, we firstly show (2.60) for $m=\mu$: We only have to refer equation (2.55) when $m=\mu$, and replace $\phi_{h'}$ by ϕ_x: We derive an equation:

$$\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_\mu}\sum_{x\in X^\mu} a_x C^{x,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\hat{\phi}_h,\dots,\phi_u,\phi_{u+1}) = \\
&Xdiv_{i_1}\dots Xdiv_{i_{\mu+1}}\sum_{x\in\overline{X}^{\mu+1}} a_x C^{x,i_1\dots i_{m+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\hat{\phi}_h,\dots,\phi_u,\phi_{u+1})+ \\
&\sum_{t\in T} a_t Xdiv_{i_1}\dots Xdiv_{i_\mu} C^{t,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})+ \\
&\sum_{j\in J\bigcup J'\bigcup J''} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),
\end{aligned} \tag{2.62}$$

(using generic notation of the tensor fields in the RHS).

Now, we consider the tensor fields indexed in $\overline{X}^{\mu+1}$ which have a free index in the factor $Ric_{ab}\nabla^a\phi_{h'}$[55] and we "forget" the $Xdiv$ structure of $Xdiv_b$. Therefore, we are reduced to proving (2.60) with with two additionnal features if $m=\mu$: Firstly that the tensor fields with rank $m=\mu$ *do not* have a free index in the expression $Ric_{ab}\nabla^a\tilde{\phi}_{h'}$ and also that for those tensor fields the

[55] In other words, the index $_b$ is free.

index $_b$ in that expression is contracting either against a non-special index in some curvature factor or against some index in a factor $\nabla^{(A)}\Omega_f$ with $A \geq 3$.[56]

Armed with this additionnal hypothesis for the case $m = \mu$, we will now show (2.60) for any $m \geq \mu$:

We apply the operation $Ricto\Omega$ to the Lemma hypothesis (using the notation of (2.59)), and we pick out the sublinear combination with an expression $\nabla^{(2)}_{ij}\Omega_{p+1}\nabla^i\phi_{h'}$. It follows that this expression (which we denote by E_g) vanishes separately. Thus, we derive an equation:

$$E_g = X_*div_{i_1} \dots X_*div_{i_a} \sum_{x \in X^m} a_x C^{x,i_1\dots i_a}_g(\Omega_1, \dots, \Omega_{p+1}, \phi_1, \dots, \hat{\phi}_x, \dots, \phi_u, \phi_{u+1}) \\ + \sum_{j \in J'} a_j C^j_g(\Omega_1, \dots, \Omega_{p+1}, \phi_1, \dots, \hat{\phi}_h, \dots, \phi_u, \phi_{u+1}) = 0, \tag{2.63}$$

where X_*div_i stands for the sublinear combination in $Xdiv_i$ where ∇_i is in addition not allowed to hit the expression $\nabla^{(2)}_{ij}\Omega_{p+1}\nabla^i\phi_{h'}$. Now, we formally replace the expression $\nabla^{(2)}_{ij}\Omega_{p+1}\nabla^i\phi_{h'}$ by a factor $\nabla_j Y$. We denote the resulting tensor fields and complete contractions by:

$$C^{x,i_1\dots i_a}_g(\Omega_1, \dots, \Omega_p, Y, \phi_1, \dots, \hat{\phi}_h, \dots, \phi_u, \phi_{u+1}),$$
$$C^j_g(\Omega_1, \dots, \Omega_p, Y, \phi_1, \dots, \hat{\phi}_h, \dots, \phi_u, \phi_{u+1}).$$

Then, since the above equation holds formally we derive that:

$$X_*div_{i_1} \dots X_*div_{i_a} \sum_{x \in X} a_x C^{x,i_1\dots i_a}_g(\Omega_1, \dots, \Omega_p, Y, \phi_1, \dots, \hat{\phi}_h, \dots, \phi_u, \phi_{u+1}) + \\ \sum_{j \in J'} a_j C^j_g(\Omega_1, \dots, \Omega_p, Y, \phi_1, \dots, \hat{\phi}_h, \dots, \phi_u, \phi_{u+1}) = 0. \tag{2.64}$$

X_*div_i in this setting stands for the sublinear combination in $Xdiv_i$ where ∇_i is not allowed to hit the factor ∇Y.

But then (subject to certain exceptions which we explain below) applying 4.6 in [6]),[57] (or Lemma 4.7 in [6] if $\sigma = 3$)[58] to the above,[59] we derive that

[56](The last property follows since ∇^b has arisen by "forgetting" an $Xdiv$).

[57]In particular, the exceptions are when there are tensor fields of minimum rank in (2.64) that fall under one of the "forbidden cases" of that Lemma. The derivation of (2.60) in that case will be discussed below.

[58]Notice that by the conventions above this Lemma can be applied since the tensor fields of minimum rank do not have a free index in ∇Y.

[59]Equation (2.64) formally falls under the inductive assumptions of these Lemmas, since we have reduced the weight.

there is a linear combination of $(m+1)$-tensor fields,

$$\sum_{q\in Q} a_q C^{q,i_1\dots i_{m+1}}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\hat{\phi}_x,\dots,\phi_u,\phi_{u+1}),$$

just like the ones indexed in X^m only with another free index, so that:

$$\begin{gathered}\sum_{x\in X} a_x C^{x,i_1\dots i_m}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\hat{\phi}_x,\dots,\phi_u,\phi_{u+1})\nabla_{i_1}\upsilon\dots\nabla_{i_m}\upsilon-\sum_{q\in Q} a_q \\ X_*div_{i_{m+1}}C^{q,i_1\dots i_{m+1}}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\hat{\phi}_x,\dots,\phi_u,\phi_{u+1})\nabla_{i_1}\upsilon\dots\nabla_{i_m}\upsilon \\ +\sum_{j\in J'} a_j C^{j,i_1\dots i_m}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\hat{\phi}_x,\dots,\phi_u,\phi_{u+1})\nabla_{i_1}\upsilon\dots\nabla_{i_m}\upsilon=0.\end{gathered} \tag{2.65}$$

The Exceptions: In the exceptional cases, we apply Lemma 4.10 in [6] to (2.64) to derive (2.60) directly. (Notice that this Lemma can be applied since $m>\mu$ in this case; this is because of the additionnal hypothesis in the case $m=\mu$ which ensures that we do not fall under the forbidden case when $m=\mu$).

Derivation of (2.60) from (2.65):[60] Now, formally replace $\nabla_a Y$ by an expression $Ric_{ia}\nabla^a\phi_{h'}$ in the above; we thus again obtain an equation:

$$\begin{gathered}\sum_{x\in X} a_x C^{x,i_1\dots i_m}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\hat{\phi}_x,\dots,\phi_u,\phi_{u+1})\nabla_{i_1}\upsilon\dots\nabla_{i_m}\upsilon- \\ \sum_{q\in Q} a_q X_*div_{i_{m+1}}C^{q,i_1\dots i_{m+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\hat{\phi}_x,\dots,\phi_u,\phi_{u+1})\nabla_{i_1}\upsilon\dots\nabla_{i_m}\upsilon \\ +\sum_{j\in J'} a_j C^{j,i_1\dots i_m}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\hat{\phi}_x,\dots,\phi_u,\phi_{u+1})\nabla_{i_1}\upsilon\dots\nabla_{i_m}\upsilon=0.\end{gathered} \tag{2.66}$$

Observe that making the X_*div into and $Xdiv$ introduces tensor fields with a factor $\nabla Ric_{ij}\nabla^i\phi_{h'}$; where we may then apply the second Bianchi identity to this expression and make the factor $\nabla\phi_{h'}$ contract against the derivative index in ∇Ric. We obtain correction terms that are in the form:

$$\sum_{t\in T} a_t Xdiv_{i_1}\dots Xdiv_{i_a}C^{t,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).$$

Thus, replacing the $\nabla\upsilon$s by $Xdiv$s (see the last Lemma in the Appendix in [3]), we obtain our desired (2.60). □

[60] In the non-exceptional cases.

3 An analysis of the sublinear combination $CurvTrans[L_g]$.

3.1 Brief outline of this section: How to "get rid" of the terms with $\sigma + u$ factors in (6.1), modulo correction terms we can control.

Let us recapitulate to recall our main achievements so far and to outline how our argument will proceed: We have set out to prove Lemmas 1.1, 1.2 (and, eventually Lemma 1.3, under the inductive assumption of Proposition 1.1, along with all the Corollaries and Lemmas that the inductive assumption of Proposition 1.1 implies. The main assumption for all these Lemmas is equation (2.3), whose left hand side we denote by $L_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)$ (or just L_g, for short).

In Lemma 2.1 we showed that the sublinear combination $Image^{1,+}_{\phi_{u+1}}[L_g]$ must vanish separately (modulo complete contractions that we may ignore). The equation (2.8) is the main assumption for this section. In equation (6.1) we broke up $Image^{1,+}_{\phi_{u+1}}[L_g]$ into three sublinear combinations $CurvTrans[L_g]$, $LC[L_g]$, $W[L_g]$ which we will study separately in the next few subsections (the reader may wish to recall these three sublinear combinations now).

Finally, in Lemma 2.2 we showed that the sublinear combination $LC_\Phi[L_g]$ (in $LC[L_g]$) can be replaced by the right hand side of the equation in (2.2). Since that right hand side consists of generic terms that are allowed in the conclusions of the Lemmas 1.1, 1.2 and Lemma 1.3 in case A, we may interpret this result as saying that the sublinear combination $LC_\Phi[L_g]$ can be *ignored* when we study $LC[L_g]$ further down.

Synopsis of subsections 3.2, 3.3: We commence this section with a study of the sublinear combination $CurvTrans[L_g]$. The next two subsections (3.3 and 3.2) are devoted to that goal. Our analysis will proceed as follows: We will firstly seek to understand the sublinear combination $CurvTrans[L_g]$ as it arises from the application of the formula (2.1). We will observe that the terms we obtain in $CurvTrans[L_g]$ can be grouped up into a few sublinear combinations, defined by certain algebraic properties. After we do this grouping, we will apply the curvature identity,

$$\nabla^{(2)}_{ab} X_c - \nabla^{(2)}_{ba} X_c = R_{abdc} X^d, \tag{3.1}$$

which will introduce corrections terms of length $\sigma + u + 1$, some of which will be important in deriving our Lemmas 1.1, 1.2, 1.3 (in particular the sublinear combinations in $Leftover[\dots]$ will be the important ones), and many will be generic terms (i.e. generic terms allowed in the conclusions of our three Lemmas). Finally, after this analysis and the algebraic manipulations, we will still be left with sublinear combinations in $CurvTrans[L_g]$ of length $\sigma + u$. Roughly speaking, these sublinear combinations will either be linear combinations of iterated $Xdiv$'s with high enough rank, or they will be terms that are simply subsequent to $pre\vec{\kappa}^{+}_{simp}$. We will then show that these sublinear combinations

can be *re-written* as linear combinations of $Xdiv$'s of tensors fields *with* $\sigma+u+1$ *factors*, of the general type that is allowed in the conclusions of our Lemmas.

Notational conventions: Now, abusing notation, we will denote $Image^{1,+}_{\phi_{u+1}}[L_g]$ by $Image^{1}_{\phi_{u+1}}[L_g]$ for the rest of this section. Furthermore, we recall that in the setting of Lemmas 1.1 and 1.2 the *selected factor* discussed in the definition of $Image^{1,+}_{\phi_{u+1}}[L_g]$ is *always* the crucial factor, defined, in the statements of Lemmas 1.1 and 1.2. On the other hand, in the setting of 1.3, we have declared that the selected factor is some factor (or set of factors) that we pick once and for all; it does not have to be the crucial factor. Recall that $Image^{1,+}_{\phi_{u+1}}[L_g]$ has been defined in definition 2.2–this sublinear combination depends on the choice of selected factor. Therefore, in the next subsections, we will sometimes be making distinctions when we discuss the sublinear combination $CurvTrans[L_g]$; these distinctions will depend on which of the Lemmas 1.1, 1.2 or 1.3 we are proving.

3.2 A study of the sublinear combination $CurvTrans[L_g]$ in the setting of Lemmas 1.2 and 1.3 (when the selected factor is in the form $\nabla^{(m)}R_{ijkl}$).

Let us firstly recall that in the setting of Lemma 1.2 the notions of "selected"and "crucial" factor coincide. In the setting of Lemma 1.3 they need not coincide. Furthermore, in the setting of Lemma 1.2, we will be denoting by $Free(Max)$ the number of free indices in the crucial factor in the tensor fields $C^{l,i_1\dots i_\mu}_g, l\in \bigcup_{z\in Z'_{Max}} L^z$ (see the statement of Lemma 1.2).[61] Whenever we make a claim regarding 1.3 in this subsection, we will be assuming that the selected factor is in the from $\nabla^{(m)}R_{ijkl}$.

In this case, we recall that

$$CurvTrans[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

stands for the sublinear combination in

$$Image^{1}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

[62] that consists of complete contractions with length $\sigma+u$, with no internal contractions that arise by replacing the (one of the) selected factor(s) $\nabla^{(m)}R_{ijkl}$ by one of the four linear expressions $\nabla^{(m+2)}_{r_1\dots r_m il}\phi_{u+1}g_{jk}$ etc on the right hand side of (2.1), provided no internal contraction arises in that way. We will be treating the function $\nabla^{(A)}\phi_{u+1}$ as a function $\nabla^{(A)}\Omega_{p+1}$ in this subsection.

[61]Recall that by definition, if the μ-tensorf fields of maximal refined double character in (1.6) have s special free indices in the crucial factor $\nabla^{(m)}R_{ijkl}$ ($s=1$ or $s=2$) then all other μ-tensor fields in the assumption of Lemma 1.2 will have at most $Free(Max)$ free indices in any factor $\nabla^{(m)}R_{ijkl}$ that contains s special free indices.

[62]Recall that we are denoting $Image^{1,+}_{\phi_{u+1}}[\dots]$ by $Image^{1}_{\phi_{u+1}}[\dots]$, abusing notation.

In this setting, a complete contraction in

$$CurvTrans[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

will be called *extra acceptable* if it is acceptable (see the discussion after (1.5) in [6]) *and* in addition it has all the factors $\nabla\phi_{g_i}$ contracting against the factor $\nabla^{(m+2)}\phi_{u+1}$,[63] *and* the two rightmost indices in $\nabla^{(A)}\phi_{u+1}$ are not contracting against any factor $\nabla\phi_h$. We straightforwardly observe that all the complete contractions in each sublinear combination

$$CurvTrans[Xdiv_{i_1}\dots Xdiv_{i_a}C_g^{l,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] \tag{3.2}$$

are extra acceptable.

As before, we will be using the equation:

$$\begin{split}&\sum_{l\in L} a_l CurvTrans[Xdiv_{i_1}\dots Xdiv_{i_a}C_g^{l,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\&\sum_{j\in J} a_j CurvTrans[C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=0,\end{split} \tag{3.3}$$

which holds modulo complete contractions of length $\geq \sigma+u+1$.

Now, we separately study the sublinear combinations in the left hand side of the above.

We start with the sublinear combinations $CurvTrans[C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ for each $j\in J$.

Let us introduce some notation. We will denote by

$$\sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)$$

a generic linear combination of complete contractions in the form (2.15) with a weak character $Weak(pre\vec{\kappa}^+_{simp})$ and with at least one factor $\nabla\phi_f$, $f\in Def(\vec{\kappa}_{simp})$ contracting against a derivative index in some factor $\nabla^{(p)}R_{ijkl}$. We then straightforwardly observe that:

$$CurvTrans[\sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] = \sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u). \tag{3.4}$$

Next, we proceed to carefully study the sublinear combinations (3.2) in (3.3). We will need to introduce some further notational conventions.

[63] Here $\{\nabla\phi_{g_i}\}$ stands for the set of terms $\nabla\phi_h$ that are contracting against the selected factor $\nabla^{(m)}R_{ijkl}$ in $\vec{\kappa}_{simp}$.

Definition 3.1 *For this entire subsection, we denote by*

$$\sum_{p\in P} a_p C^{p,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}$$

a generic linear combination of acceptable tensor fields with length $\sigma+u+1$, and $a\ge\mu$ and with a u-simple character $\vec{\kappa}_{simp}$ and a weak $(u+1)$-character $Weak(\vec{\kappa}^+_{simp})$.

Furthermore, we denote by

$$\sum_{u\in U} a_u Xdiv_{i_1}\dots Xdiv_{i_a} C^{u,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)$$

a generic linear combination of $Xdiv$s of extra acceptable a-tensor fields ($a\ge\mu$) with the following features: The tensor fields $C^{u,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)$ must have of length $\sigma+u$, be in the form (2.15) and have a u-simple character $pre\vec{\kappa}^+_{simp}$, where all the factors $\nabla\phi_{g_1},\dots,\nabla\phi_{g_z}$ are precisely those $\nabla\phi$'s that are contracting against the first z indices in the factor $\nabla^{(A)}\phi_{u+1}$ with $A\ge z+2$. Furthermore, we require that if $a=\mu$ then either at least one of the free indices $i_1,\dots,i_\mu$ is a derivative index (and moreover if it belongs to a factor $\nabla^{(B)}\Omega_h$ then $B\ge 3$), or none of these indices is a special index in a factor $S_*\nabla^{(\nu)}R_{ijkl}$.

Our next definitions will be only for the setting of Lemma 1.2.

We will denote by

$$\sum_{t\in T} a_t C^{t,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}$$

a generic linear combination of acceptable $(\mu-1)$-tensor fields of length $\sigma+u+1$ with $(u+1)$-simple character $\vec{\kappa}^+_{simp}$, for which either the selected (=crucial=critical) factor contains fewer than $Free(Max)-1$ free indices, or it contains exactly $Free(Max)-1$ free indices but its refined double character is doubly subsequent to each $\vec{L^z}'$, $z\in Z'_{Max}$.[64]

In addition (again only in the setting of Lemma 1.2) we denote by

$$\sum_{u\in U_1} a_u C^{u,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)$$

a generic linear combination of extra acceptable $(\mu-1)$ tensor fields in the form (2.15), of length $\sigma+u$ with a simple character $pre\vec{\kappa}^+_{simp}$, with the extra property that the factor $\nabla^{(A)}\phi_{u+1}$ has fewer than $Free(Max)-1$ free indices. We also require that none of the free indices are special indices in a factor $S_*\nabla^{(\nu)}R_{ijkl}$.

Moreover, in the case where the maximal refined double characters $\vec{L^z}'$, $z\in Z'_{Max}$ have two internal free indices in the crucial factor, we denote by

$$\sum_{u\in U_2} a_u C^{u,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)$$

[64] In other words, in the statements of Lemma 1.2 this corresponds to a generic linear combination $\sum_{\nu\in N} a_\nu C^{\nu,i_1\dots,i_{\mu-1}i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_\mu}\phi_{u+1}$.

a generic linear combination of extra acceptable $(\mu-1)$ tensor fields in the form (2.15) with a simple character $pre\vec{\kappa}^+_{simp}$ and with the feature that it has $Free(Max)-1$ free indices in the factor $\nabla^{(A)}\phi_{u+1}$ and with the additional property that one of the free indices that belong to the factor $\nabla^{(A)}_{r_1\dots r_A}\phi_{u+1}$ is the last index ${}_{r_A}$. We also require that none of the free indices are special indices in a factor $S_*\nabla^{(\nu)}R_{ijkl}$.

Armed with this definition, we may now study the sublinear combination (3.2) in detail.

We recall a few notational conventions we have made in the setting of Lemma 1.2:

Recall that in the setting of Lemma 1.2, the crucial factor(s) in each $C^{l,i_1\dots i_\mu}$, $l\in L^z, z\in Z'_{Max}$ will all have either two, one or no internal free indices.

If the maximal refined double characters $\vec{L^z}, z\in Z'_{Max}$ have one or two internal free indices belonging to the (a) crucial factor, we recall that we have denoted by $I_{*,l}\subset I_l$ the set of all internal free indices that belong to a crucial factor. Furthermore, if there are two such free indices, we have declared that in the (each) crucial factor of the form $T=\nabla^{(m)}R_{ijkl}$, the internal free indices will be the indices ${}_i, {}_k$. In that setting, we may then assume wlog that the indices ${}_{i_1}, {}_{i_3},\dots, {}_{i_{2k_l+1}}\in I_{*,l}$ are the indices ${}_i$ in the crucial factors. Also in this case we may assume wlog that if k is odd then ${}_k$ and ${}_{k+1}$ belong to the same crucial factor. We then claim:

Lemma 3.1 *In the setting of Lemma 1.2, if the tensor fields $C^{l,i_1\dots i_\mu}_g, l\in L^z, z\in Z'_{Max}$ have two internal free indices in the crucial factor(s) then:*

$$\begin{aligned}
&\sum_{l\in L_\mu} a_l CurvTrans[Xdiv_{i_1}\dots Xdiv_{i_\mu}C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&\sum_{z\in Z'_{Max}}\sum_{l\in L^z} a_l\sum_{h=0}^{k_l} Xdiv_{i_1}\dots X\hat{div}_{i_{2h+1}}\dots Xdiv_{i_\mu}\\
&C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\\
&+\sum_{u\in U} a_u Xdiv_{i_1}\dots Xdiv_{i_a}C^{u,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+\\
&\sum_{u\in U_1} a_u Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}}C^{u,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+\\
&\sum_{u\in U_2} a_u Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}}C^{u,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+\\
&\sum_{t\in T} a_t Xdiv_{i_2}\dots Xdiv_{i_\mu}C^{t,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}.
\end{aligned}\tag{3.5}$$

Next claim: Again in the setting of Lemma 1.2, if the tensor fields $C^{l,i_1\dots i_\mu}_g, l\in L^z, z\in Z'_{Max}$ have one internal free index in the crucial factor(s) then:

$$\sum_{l\in L_\mu} a_l CurvTrans[Xdiv_{i_1}\dots Xdiv_{i_\mu} C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] =$$
$$\sum_{z\in Z'_{Max}}\sum_{l\in L^z} a_l \sum_{i_h\in I_{*,l}} Xdiv_{i_1}\dots X\hat{div}_{i_h}\dots Xdiv_{i_\mu} C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$$
$$\nabla_{i_h}\phi_{u+1} + \sum_{u\in U} a_u Xdiv_{i_1}\dots Xdiv_{i_\mu} C^{u,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+$$
$$\sum_{t\in T} a_t Xdiv_{i_2}\dots Xdiv_{i_\mu} C^{t,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}. \tag{3.6}$$

In the setting of Lemma 1.3 and also in the setting of Lemma 1.2 if the tensor fields $C^{l,i_1\dots i_\mu}_g, l\in L^z, z\in Z'_{Max}$ *have no internal free index in the crucial factor(s) then:*

$$\begin{aligned}&\sum_{l\in L_\mu} a_l CurvTrans[Xdiv_{i_1}\dots Xdiv_{i_\mu} C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\\ &=\sum_{u\in U} a_u Xdiv_{i_1}\dots Xdiv_{i_\mu} C^{u,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u).\end{aligned} \tag{3.7}$$

Moreover, for both Lemmas 1.2 and 1.3:

$$\begin{aligned}&\sum_{l\in (L\setminus L_\mu)} a_l CurvTrans[Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\\ &=\sum_{p\in P} a_p Xdiv_{i_1}\dots Xdiv_{i_a} C^{p,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\ &\sum_{u\in U} a_u Xdiv_{i_1}\dots Xdiv_{i_a} C^{u,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u).\end{aligned} \tag{3.8}$$

Proof of Lemma 3.1: The proof just follows by applying the transformation law (2.1). For the first two equations in the Lemma, we "complete the divergence" to get the terms on the first two lines of the right hand sides.[65] Also, for the first two equations, the proof of our claim also relies on the definition of *maximal* refined double characters. *Note:* For (3.6) we also use the fact that (1.6) does not fall under the "special cases" outlined at the very end of the

[65]We explain the notion of "completing the divergence": We observe that we obtain terms in $CurvTrans[Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g]$ which are in the form $X_*div_{i_1}Xdiv_{i_2}\dots Xdiv_{i_\mu}C^{*,i_1\dots i_\mu}_g$, where the free index $_{i_1}$ does not belong to the factor $\nabla^{(B)}\phi_{u+1}$, and $X_*div_{i_1}$ means that ∇^{i_1} is not allowed to hit the factor $\nabla^{(B)}\phi_{u+1}$. Then adding and subtracting a term $Hitdiv_{i_1}Xdiv_{i_2}\dots Xdiv_{i_\mu}C^{*,i_1\dots i_\mu}_g$ ($Hitdiv_{i_1}$ means that we force ∇^{i_1} to hit the factor $\nabla^{(B)}\phi_{u+1}$), we obtain the terms of length $\sigma+u+1$ in (3.5), (3.6) (when we subtract the term in question), by also applying the curvature identity (3.1).

introduction. $\Box$

In conclusion, we have shown that in the setting of Lemma 1.2 when there are two internal free indices in the crucial factor in the tensor fields $C^{l,i_1\dots i_\mu}_g$, $L\in L^z, z\in Z'_{Max}$ we will have:

$$\begin{aligned}
&CurvTrans[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&\sum_{z\in Z'_{Max}}\sum_{l\in L^z}a_l\sum_{h=0}^{k_l}Xdiv_{i_1}\dots X\hat{div}_{i_{2h+1}}\dots Xdiv_{i_\mu}C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\\
&+\nabla_{i_1}\phi_{u+1}\sum_{u\in U}a_uXdiv_{i_1}\dots Xdiv_{i_a}C^{u,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+\\
&\sum_{u\in U_1}a_uXdiv_{i_1}\dots Xdiv_{i_{\mu-1}}C^{u,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\\
&+\sum_{u\in U_2}a_uXdiv_{i_1}\dots Xdiv_{i_{\mu-1}}C^{u,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\\
&+\sum_{j\in J}a_jC^j_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+\\
&\sum_{p\in P}a_pXdiv_{i_1}\dots Xdiv_{i_a}C^{p,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{t\in T}a_tXdiv_{i_1}\dots Xdiv_{i_{\mu-1}}C^{t,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_\mu}\phi_{u+1}.
\end{aligned}\tag{3.9}$$

Also, in the setting of Lemma 1.2 when there is one internal free index in the crucial factor in the tensor fields $C^{l,i_1\dots i_\mu}_g$, $l\in L^z, z\in Z'_{Max}$ we will have:

$$\begin{aligned}
&CurvTrans[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&\sum_{z\in Z'_{Max}}\sum_{l\in L^z}a_l\sum_{i_h\in I_{*,l}}Xdiv_{i_1}\dots X\hat{div}_{i_h}\dots Xdiv_{i_\mu}C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\\
&+\nabla_{i_h}\phi_{u+1}\sum_{u\in U}a_uXdiv_{i_1}\dots Xdiv_{i_a}C^{u,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\\
&+\sum_{u\in U_1}a_uXdiv_{i_1}\dots Xdiv_{i_{\mu-1}}C^{u,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+\\
&\sum_{j\in J}a_jC^j_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+\\
&\sum_{p\in P}a_pXdiv_{i_1}\dots Xdiv_{i_a}C^{p,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{t\in T}a_tXdiv_{i_1}\dots Xdiv_{i_{\mu-1}}C^{t,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_\mu}\phi_{u+1},
\end{aligned}\tag{3.10}$$

and lastly in the setting of Lemma 1.3 we will have:

$$
\begin{aligned}
&CurvTrans[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] = \\
&\sum_{u\in U} a_u Xdiv_{i_1}\dots Xdiv_{i_a} C^{u,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+ \\
&+\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+ \\
&\sum_{p\in P} a_p Xdiv_{i_1}\dots Xdiv_{i_a} C^{p,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}.
\end{aligned}
\tag{3.11}
$$

We then make three claims, for each of the three subcases above. We are interested in "getting rid of" the sublinear combinations of length $\sigma+u$ that we have been left with in $CurvTrans[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$. We first consider the setting of Lemma 1.2 and there are tensor fields indexed in $L_\mu\subset L$ with two internal free indices in the crucial factor $\nabla^{(m)}R_{ijkl}$; call this the first subcase. We claim:

Lemma 3.2 *Consider the setting of Lemma 1.2 when there are tensor fields indexed in $L_\mu\subset L$ with two free indices in the crucial factor $\nabla^{(m)}R_{ijkl}$. Then, refer to (3.9). By virtue of our inductive assumption on Proposition 1.1, we claim that the sublinear combination of length $\sigma+u$ in (3.9) will be equal to:*

$$
\begin{aligned}
&\sum_{u\in U} a_u Xdiv_{i_1}\dots Xdiv_{i_a} C^{u,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+ \\
&\sum_{u\in U_1} a_u Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} C^{u,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+ \\
&\sum_{u\in U_2} a_u Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} C^{u,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u) \\
&+\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u) = \\
&\sum_{r\in R_\alpha} a_r Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} C^{r,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}) \\
&+\sum_{r\in R_\beta} a_r Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} C^{r,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}) \\
&+\sum_{r\in R_\gamma} a_r Xdiv_{i_1}\dots Xdiv_{i_a} C^{r,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})+ \\
&\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1});
\end{aligned}
\tag{3.12}
$$

here the $(\mu-1)$-tensor fields indexed in R_α have a $(u+1)$-simple character $\vec{\kappa}^+_{simp}$, and also have $Free(Max)-1$ free indices belonging to the crucial factor

$\nabla^{(m)}_{r_1\dots r_m}R_{ijkl}$ *and moreover all of them are of the form* ${}_{r_1},\dots,{}_{r_m},{}_j$ *and furthermore* $\nabla\phi_{u+1}$ *is contracting against the index* ${}_i$ *(so in particular they are doubly subsequent to all* $\vec{L^z}'$, $z\in Z'_{Max}$*).*

Also, the $(\mu-1)$*-tensor fields that are indexed in* R_β *have a refined double character that is doubly subsequent to each* $\vec{L^z}'$, $z\in Z'_{Max}$*. (In particular they have fewer than* $Free(Max)-1$ *free indices in the crucial factor). Finally, each* a*-tensor field* $(a\geq\mu)$ *indexed in* R_γ *has a* u*-simple character* $\vec{\kappa}_{simp}$ *and a weak* $(u+1)$*-character* $Weak(\vec{\kappa}^+_{simp})$*. Lastly, each* C^j *has length* $\sigma+u+1$ *and is simply subsequent to* $\vec{\kappa}^+_{simp}$*.*

Next, we consider the setting of Lemma 1.2 where there is one internal free index in the tensor fields $C^{l,i_1\dots i_\mu}_g, l\in L^z, z\in Z'_{Max}$ (call this the *second subcase*). We also consider the setting of Lemma 1.3 (where there are no internal free indices in any factor $\nabla^{(m)}R_{ijkl}$ in the μ-tensor fields we are considering; call this the *third subcase*).

Lemma 3.3 *By virtue of our inductive assumption on Proposition 1.1, in both (second and third) subcases above we claim that the sublinear combination of length* $\sigma+u$ *that we have been left with in* $CurvTrans[L_g]$ *(see (3.10) and (3.11)) can be written as:*

$$
\begin{aligned}
&\sum_{u\in U} a_u Xdiv_{i_1}\dots Xdiv_{i_a} C^{u,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+\\
&\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)=\\
&(\sum_{r\in R_\beta} a_r Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} C^{r,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_{u+1}))+\\
&\sum_{r\in R_\gamma} a_r Xdiv_{i_1}\dots Xdiv_{i_a} C^{r,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})+\\
&\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1});
\end{aligned}
\tag{3.13}
$$

The $(\mu-1)$*-tensor fields that are indexed in* R_β *arise only in the second subcase and have a (refined) double character that is subsequent to each* $\vec{L^z}'$, $z\in Z'_{Max}$*. The tensor fields in* R_β, R_γ *are as above.*

Proof of Lemmas 3.2 and 3.3:

We will prove Lemma 3.2. We will then indicate how Lemma 3.3 follows by the same argument.

By virtue of the equation (6.1) we have that:

$$
\begin{aligned}
&\sum_{u\in U} a_u Xdiv_{i_1}\dots Xdiv_{i_a} C^{u,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+\\
&\sum_{u\in U_1} a_u Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} C^{u,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+\\
&\sum_{u\in U_2} a_u Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} C^{u,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\\
&+\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)=0,
\end{aligned}
\tag{3.14}
$$

modulo contractions of length $\geq \sigma+u+1$.

We focus on the left hand side linear combination in (3.14), where we treat the function ϕ_{u+1} as a function Ω_{p+1}, and we apply the eraser to the factors $\nabla\phi_g$ that are contracting against $\nabla^{(A)}\phi_{u+1}$. (We will be applying this operation in order to apply the inductive assumption of Corollary 1 in [6] on various occasions below; after we have applied our Corollary, we will then re-introduce the factors $\nabla\phi_g$ that we erased.[66]) We observe that the tensor fields of length $\sigma+u$ that we obtain via this operation are acceptable, and will all have the same simple character which we denote by $\vec{\kappa}_{simp}$. Furthermore, the complete contractions $C^j_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)$ that arise after applying the eraser will be subsequent to the simple character $\vec{\kappa}_{simp}$.

Thus, we can apply our inductive assumption on Corollary 1 in [6][67] to the left hand side of (3.14) (to which we have applied the eraser). We conclude that there is a linear combination of acceptable μ-tensor fields (indexed in T below) with a simple character $\vec{\kappa}_{simp}$ so that:

$$
\begin{aligned}
&\sum_{u\in U_1} a_u C^{u,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon+\\
&\sum_{u\in U_2} a_u C^{u,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon-\\
&\sum_{t\in T} a_u Xdiv_{i_\mu} C^{t,i_1\dots i_{\mu-1}i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon=\\
&\sum_{f\in F} a_f C^{f,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon,
\end{aligned}
\tag{3.15}
$$

modulo complete contractions of length $\geq \sigma+u+\mu$. Here each $C^{f,i_1\dots i_{\mu-1}}_g$ is simply subsequent to $\vec{\kappa}_{simp}$. Now, we index in $T_2\subset T$ the tensor fields

[66] By abuse of notation we will sometimes use $\vec{\kappa}_{simp}$ to also denote the simple character with the factors $\nabla\phi_g$ put back in.

[67] Since the tensor fields indexed in U_1 have no special free indices in factors $S_*\nabla^{(\nu)}R_{ijkl}$, there is no danger of falling under a "forbidden case".

with precisely $Free(Max)-1$ factors $\nabla\upsilon$ contracting against $\nabla^{(A)}\phi_{u+1}$ and in $T_1 \subset T$ the tensor fields with fewer than $Free(Max)-1$ factors $\nabla\upsilon$ contracting against $\nabla^{(A)}\phi_{u+1}$. Since the above holds formally, we then derive two equations: Firstly:

$$\sum_{u\in U_2} a_u C^{u,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon -$$
$$\sum_{t\in T_2} a_u Xdiv_{i_\mu} C^{t,i_1\dots i_{\mu-1}i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon =$$
$$\sum_{f\in F} a_f C^{f,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon, \tag{3.16}$$

modulo complete contractions of length $\geq \sigma+u+\mu$. Moreover, we may assume with no loss of generality that for each of the tensor fields $C^{t,i_1\dots i_{\mu-1}i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)$ above, one of the free indices ${}_{i_1},\dots,{}_{i_{\mu-1}}$ that belongs to the factor $\nabla^{(A)}\phi_{u+1} = \nabla^{(A)}\Omega_{p+1}$ is the last index ${}_{r_A}$ in that factor (as is also the case of the contractions indexed in U_2).

Secondly, we derive:

$$\sum_{u\in U_1} a_u C^{u,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon -$$
$$\sum_{t\in T_1} a_u Xdiv_{i_\mu} C^{t,i_1\dots i_{\mu-1}i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon =$$
$$\sum_{f\in F} a_f C^{f,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon, \tag{3.17}$$

modulo complete contractions of length $\geq \sigma+u+\mu$.

Now, we seek to use the fact that the above equations hold formally to derive information about the correction terms of length $\sigma+\mu+u$ in (3.16) and (3.17). In (3.16) we use the fact that the index ${}_{i_1}$ is the last index in the factor $\nabla^{(A)}\phi_{u+1}$. We may then assume (using the eraser) that in the permutations by which we make the left hand side of (3.16) formally zero, the last index in the factor $\nabla^{(A)}\phi_{u+1}$ (which is contracting against the factor $\nabla\upsilon$) is not permuted. We conclude that we can write out:

$$
\begin{aligned}
&\sum_{u\in U_2} a_u C^{u,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon-\\
&\sum_{t\in T_2} a_u Xdiv_{i_\mu} C^{t,i_1\dots i_{\mu-1}i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon=\\
&\sum_{r\in R_\alpha} a_r C^{r,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon\\
&+\sum_{f\in F} a_f C^{f,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon\\
&+\sum_{z\in Z} a_z C^{z,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon,
\end{aligned}
\tag{3.18}
$$

where each tensor field indexed in Z has length $\sigma+u+1$ and a factor $\nabla^{(b)}\phi_{u+1}$, $b\geq 2$. The above holds modulo correction terms of length greater than the RHS.

By a similar argument, we derive that we can write:

$$
\begin{aligned}
&\sum_{u\in U_1} a_u C^{u,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon-\\
&\sum_{t\in T_1} a_u Xdiv_{i_\mu} C^{t,i_1\dots i_{\mu-1}i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon=\\
&\sum_{f\in F} a_f C^{f,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon+\\
&\sum_{r\in R_\beta} a_r C^{r,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon\\
&+\sum_{z\in Z} a_z C^{z,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon.
\end{aligned}
\tag{3.19}
$$

The above holds modulo correction terms of length greater than the RHS.

Therefore, we make the $\nabla\upsilon$'s into $Xdiv$'s (by applying the last Lemma in the Appendix of [3]) in the above two equations we deduce that:

$$\sum_{u\in U_2} a_u Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} C^{u,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)-$$
$$\sum_{t\in T_2} a_u Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} Xdiv_{i_\mu} C^{t,i_1\dots i_{\mu-1}i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)$$
$$=\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+$$
$$\sum_{r\in R_\alpha} a_r Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} C^{r,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$$
$$+\sum_{z\in Z} a_z C^{z,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1}), \tag{3.20}$$

(modulo complete contractions longer than the RHS) and also that:

$$\sum_{u\in U_1} a_u Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} C^{u,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)-$$
$$\sum_{t\in T_1} a_u Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} Xdiv_{i_\mu} C^{t,i_1\dots i_{\mu-1}i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)$$
$$=\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+$$
$$\sum_{r\in R_\beta} a_r Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} C^{r,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$$
$$+\sum_{z\in Z} a_z C^{z,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1}), \tag{3.21}$$

(modulo complete contractions longer than the RHS) where here we have added the "missing" factors $\nabla\phi_g, g\in(\vec{\kappa})_1$ (recall $(\vec{\kappa}_1)$ stands for the set of numbers g for which $\nabla\phi_g$ is contracting against $\nabla^{(A)}\phi_{u+1}$ in $pre\vec{\kappa}^+_{simp}$) onto the factor $\nabla^{(m)}\phi_{u+1}$ (which has $M\ge 2$ for all the tensor fields) for all the tensor fields and complete contractions above.

In view of these equations, we deduce that we may assume $U_1\bigcup U_2=\emptyset$ in (3.12). Then, in order to show our Lemma 3.2, we only have to show that we can write:

$$
\begin{aligned}
&\sum_{u\in U} a_u Xdiv_{i_1}\dots Xdiv_{i_a} C^{u,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+\\
&\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)=\\
&\sum_{r\in R_\gamma} a_r Xdiv_{i_1}\dots Xdiv_{i_a} C^{r,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})+\\
&\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})\\
&+\sum_{z\in Z} a_z C^{z,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1});
\end{aligned}
\tag{3.22}
$$

(recall that by definition $a\ge\mu$).

In order to see this, we firstly again apply the eraser to the factors $\nabla\phi_g, g\in(\vec{\kappa}_1)$ in (3.12). Then, we pick out the tensor fields $C^{u,i_1\dots i_a}_g$ with the smallest number $a=\delta$ of free indices, where $\delta\ge\mu$. We suppose they are indexed in $U_\delta\subset U$. We will then apply Corollary 1 in [3] to the above, but first we will make a small note regarding the potential appearance of terms in one of the "forbidden forms". By the definition of $\sum_{u\in U}\dots$, the only way that terms in (3.22) can be "forbidden" for Corollary 1 in [6] is if they have $\delta>\mu$. Thus in that case, we apply the "weaker version" of the Proposition 1.1, from the Appendix in [6]; the correction terms that we obtain are of the form we require.

Now, in the remaining cases where no tensor field appearing in U_δ is "forbidden", we use our inductive assumption of Corollary 1 in [6] and deduce that there exists some linear combination of acceptable $(\delta+1)$-tensor fields (indexed in T_δ below with simple character $\vec{\overline{\kappa}}$, so that:

$$
\begin{aligned}
&\sum_{u\in U_\delta} a_u C^{u,i_1\dots i_\delta}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\delta}\upsilon-\\
&Xdiv_{i_{\delta+1}}\sum_{t\in T_\delta} a_t C^{t,i_1\dots i_\delta,i_{\delta+1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\delta}\upsilon+\\
&\sum_{f\in F} a_f C^{f,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon=0,
\end{aligned}
\tag{3.23}
$$

modulo complete contractions of length $\ge\sigma+u+1$. Thus, since the above must hold formally, we deduce that:

$$\sum_{u\in U_\delta} a_u C^{u,i_1\dots i_\delta}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\delta}\upsilon-$$
$$Xdiv_{i_{\delta+1}}\sum_{t\in T_\delta} a_t C^{t,i_1\dots i_\delta,i_{\delta+1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\delta}\upsilon=$$
$$\sum_{r\in R_\gamma} a_r C^{r,i_1\dots i_\delta}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\delta}\upsilon+$$
$$\sum_{z\in Z} a_z C^{z,i_1\dots i_\delta}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})\nabla_{i_1}\upsilon\dots\nabla_{i_\delta}\upsilon, \tag{3.24}$$

modulo longer complete contractions. Hence, as before, we add the missing factors $\nabla\phi_g$ onto the factor $\nabla^{(A)}\phi_{u+1}$, $A\ge 2$ and make the $\nabla\upsilon$s into $Xdiv$s (applying the last Lemma in the Appendix of [3]) to deduce that:

$$\sum_{u\in U_\delta} a_u Xdiv_{i_1}\dots Xdiv_{i_\delta} C^{u,i_1\dots i_\delta}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)-$$
$$Xdiv_{i_1}\dots Xdiv_{i_\delta}Xdiv_{i_{\delta+1}}\sum_{t\in T_\delta} a_t C^{t,i_1\dots i_\delta,i_{\delta+1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)$$
$$=\sum_{f\in F} a_f C^{f,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon+$$
$$\sum_{r\in R_\gamma} a_r Xdiv_{i_1}\dots Xdiv_{i_\delta} C^{r,i_1\dots i_\delta}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+$$
$$\sum_{z\in Z} a_z C^{z,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1}). \tag{3.25}$$

Therefore, by iteratively repeating this step we may assume that $U=\emptyset$ and we are reduced to showing that:

$$\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)=\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$$
$$+\sum_{z\in Z} a_z C^{z,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1}). \tag{3.26}$$

But this follows easily: First of all, we pick out each factor $\nabla^{(A)}_{r_1\dots r_A}\Omega_h$ (including $\nabla^{(A)}\phi_{u+1}$) and we pull to the left all the indices that are contracting against a factor $\nabla\phi_f$. We can do this modulo introducing correction terms as in the right hand side of (3.26). So now for each C^j_g, we have that the factors $\nabla^{(A)}_{r_1\dots r_A}\Omega_h$ have the property that their indices that are contracting against factors $\nabla\phi_f$ are all pulled out to the left. Moreover, since we are dealing with

complete contractions with the same weak character, we may speak of *the set* of numbers $A(h) = \{a_1, \dots, a_{b_h}\}$, for which the factors $\nabla\phi_{a_1}, \dots, \nabla\phi_{a_{b_h}}$ are contracting against $\nabla^{(A)}\Omega_h$, for each $h, 1 \le h \le p+1$.

Now, we arbitrarily pick out an ordering for each set $A(h)$. Modulo introducing a linear combination

$$\sum_{j\in J} a_j C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_{u+1}) + \sum_{z\in Z} a_z C^z_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_{u+1}),$$

we may assume that the factors $\nabla\phi_{a_1}, \dots, \nabla\phi_{a_{b_h}}$ are contracting against the left b_h indices of $\nabla^{(A)}\Omega_h$ *in the order that we have picked.*

Now, we define $Sym[C^j_g(\Omega_1, \dots, \Omega_p, \phi_{u+1}, \phi_1, \dots, \phi_u)]$ to stand for the complete contraction that is obtained from $C^j_g(\Omega_1, \dots, \Omega_p, \phi_{u+1}, \phi_1, \dots, \phi_u)$ by symmetrizing over the indices in each factor $\nabla^{(A)}\Omega_h$ *that are not contracting against a factor* $\nabla\phi_h$.

By just applying the equation (3.1) we then deduce that:

$$\begin{aligned}
&\sum_{j\in J} a_j C^j_g(\Omega_1, \dots, \Omega_p, \phi_{u+1}, \phi_1, \dots, \phi_u) \\
&= \sum_{j\in J} a_j Sym[C^f_g(\Omega_1, \dots, \Omega_p, \phi_{u+1}, \phi_1, \dots, \phi_u)] \\
&+ \sum_{j\in J} a_j C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_{u+1}) + \sum_{z\in Z} a_z C^z_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_{u+1}).
\end{aligned} \tag{3.27}$$

Finally, we use the eraser to deduce that:

$$\begin{aligned}
&\sum_{j\in J} a_j Sym[C^j_g(\Omega_1, \dots, \Omega_p, \phi_{u+1}, \phi_1, \dots, \phi_u)] = \\
&\sum_{z\in Z} a_z C^z_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u, \phi_{u+1}).
\end{aligned} \tag{3.28}$$

$\Box$

The proof of Lemma 3.3 is entirely identical. We just do not have the sublinear combinations indexed in $U_1 \bigcup U_2$, hence we just iteratively apply (3.25) and then (3.26). $\Box$

In conclusion, in view of equations (3.9), (3.10), (3.11) and Lemmas 3.2 and 3.3, we have shown the following:

Conclusions: In the setting of Lemma 1.2 when there are two internal free indices in the selected factor $\nabla^{(m)}R_{ijkl}$ in $\vec{L^z}'$, $z \in Z'_{Max}$, we have derived an equation:

$$
\begin{aligned}
&CurvTrans[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] = \\
&\sum_{z\in Z'_{Max}}\sum_{l\in L^z} a_l \sum_{h=1}^{2k_l+1} Xdiv_{i_1}\dots X\hat{div}_{i_h}\dots Xdiv_{i_\mu} C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u) \\
&\nabla_{i_1}\phi_{u+1} + \sum_{t\in T} a_t Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} C^{t,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_\mu}\phi_{u+1} \\
&+\sum_{r\in R_\alpha} a_r Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} C^{r,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}) \\
&+\sum_{r\in R_\beta} a_r Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} C^{r,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}) \\
&\sum_{r\in R_\gamma} a_r Xdiv_{i_1}\dots Xdiv_{i_a} C^{r,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1}) + \\
&\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}) + \sum_{z\in Z} a_z C^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),
\end{aligned}
\tag{3.29}
$$

(using the notational conventions of equation (3.9) and Lemma 3.2).

In the setting of Lemma 1.2 when there is one internal free index in the selected factor $\nabla^{(m)}R_{ijkl}$ in $\vec{L^z}'$, $z\in Z'_{Max}$, we have derived an equation:

$$
\begin{aligned}
&CurvTrans[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] = \\
&\sum_{z\in Z'_{Max}}\sum_{l\in L^z} a_l \sum_{i_h\in I_{*,l}} Xdiv_{i_1}\dots X\hat{div}_{i_h}\dots Xdiv_{i_\mu} C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u) \\
&\nabla_{i_h}\phi_{u+1} + \sum_{t\in T} a_t Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} C^{t,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_\mu}\phi_{u+1} \\
&+\sum_{r\in R_\beta} a_r Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} C^{r,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}) + \\
&\sum_{r\in R_\gamma} a_r Xdiv_{i_1}\dots Xdiv_{i_a} C^{r,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1}) + \\
&\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}) + \sum_{z\in Z} a_z C^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).
\end{aligned}
\tag{3.30}
$$

In the setting of Lemma 1.3 with a selected factor $\nabla^{(m)}R_{ijkl}$, we have derived an equation:

$$
\begin{aligned}
&CurvTrans[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] = \\
&\sum_{r\in R_\gamma} a_r Xdiv_{i_1}\dots Xdiv_{i_a} C^{r,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})+ \\
&\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})+ \\
&\sum_{z\in Z} a_z C^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),
\end{aligned}
\tag{3.31}
$$

(using the notational conventions spelled out in Lemma 3.3).

3.3 A study of the sublinear combination $CurvTrans[L_g]$ in the context of Lemmas 1.1 and 1.3 (when the selected factor is of the form $S_*\nabla^{(\nu)}R_{ijkl}$).

In our study of $CurvTrans[L_g]$ it will be important to recall our inductive assumptions on Proposition 1.1. Recall that in the settings where we are inductively assuming Proposition 1.1, we may also apply Corollary 1 from [3] and Lemma 4.6 from [6].

Recall the $(u+1)$-simple character $\vec{\kappa}^+_{simp}$ (which corresponds to contractions with $\sigma+u+1$ factors), and also the $(u+1)$-simple character $pre\vec{\kappa}^+_{simp}$ that corresponds to contractions with $\sigma+u$ factors (see the paper [5] for a precise definition of simple character, and the Definition 2.1 for the definition of $pre\vec{\kappa}^+_{simp}$).

We then have denoted by:

$$\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$$

a generic linear combination of complete contractions of length $\geq \sigma+u+1$ which is simply subsequent to $\vec{\kappa}^+_{simp}$.

The aim of this subsection is to derive the equation (3.48) below. However, in order not to burden the reader with many new definitions from the outset, we will commence with some simple calculations and then introduce the necessary notation needed along the way. Thus, rather than stating the objective of this subsection from the outset, we will reach it at the end of this section as a consequence of some (seemingly unmotivated) calculations.

Now, in this subsection the selected factor is of the form $T=S_*\nabla^{(\nu)}R_{ijkl}$. We denote by $\nabla\phi_{Min}$ the factor that is contracting against the index ${}_i$ of T in $\vec{\kappa}_{simp}$. We recall that the $CurvTrans[L_g]$ stands for the sublinear combination in $Image^{1,+}_{\phi_{u+1}}[L_g]$ of complete contractions with length $\sigma+u$ and a weak character $Weak(pre\vec{\kappa}^+_{simp})$. Therefore, the factor $\nabla\phi_{Min}$ *must* be contracting against the factor $\nabla^{(A)}\phi_{u+1}$, for each complete contraction in $CurvTrans[L_g]$.

We observe that the sublinear combination in any term $Image^{1,+}_{\phi_{u+1}}[\sum_{l\in L} a_l Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ for which $\nabla\phi_{Min}$ *is* contracting against the factor $\nabla^{(A)}\phi_{u+1}$ can only arise by replacing the crucial factor $S_*\nabla^{(\nu)}_{r_1\dots r_n}R_{ijkl}$ by either $S_*\nabla^{(\nu+2)}_{r_1\dots r_\nu ik}\phi_{u+1}g_{jl}$ or $-S_*\nabla^{(\nu+2)}_{r_1\dots r_\nu il}\phi_{u+1}g_{jk}$ (here S_* again stands for symmetrization over the indices $_{r_1,\dots,r_\nu,j}$). Accordingly, we denote by $CurvTrans^I[\sum_{l\in L} a_l Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$,

$$CurvTrans^{II}[\sum_{l\in L} a_l Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

the sublinear combinations that arise by making these substitutions.

Now, we carefully study the linear combinations $CurvTrans^I[\sum_{l\in L} a_l Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$, $CurvTrans^{II}[\sum_{l\in L} a_l Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ and $CurvTrans[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$. In order to state our next claim, we introduce some notational conventions. Let the total number of factors $\nabla\phi_h$ contracting against the selected factor in $\vec{\kappa}_{simp}$ be π, and in particular let those factors be $\nabla\tilde{\phi}_{Min},\nabla\phi'_{e_1},\dots,\nabla\phi'_{e_{\pi-1}}$. Also, *in the setting of Lemma 1.1* we consider the total number of free indices in the selected factor, for each $C^{l,i_1\dots i_\mu}_g$, $l\in\bigcup_{z\in Z'_{Max}} L^z$. We denote that number by $Free(Max)$.

Definition 3.2 *Recall the simple character $pre\vec{\kappa}^+_{simp}$.*

We denote by

$$\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)$$

a generic linear combination of complete contractions of length $\sigma+u$ which are simply subsequent to $pre\vec{\kappa}^+_{simp}$. (Thus ϕ_{u+1} is regarded here as a factor Ω_h).

Moreover, we denote by $\sum_{d\in D^\sharp} a_d C^d_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)$ a generic linear combination of complete contractions with length $\sigma+u$, a weak u-simple character $Weak(pre\vec{\kappa}^+_{simp})$ and with $\nabla\phi_{Min}$ contracting against the first index in the factor $\nabla^{(P)}\phi_{u+1}$, where we additionally require that $P\geq 3$.

We will also let $\sum_{p\in P} a_p C^{p,i_1\dots i_a,i_*}_g\nabla_{i_*}\phi_{u+1}$ be a generic linear combination of a-tensor fields ($a\geq\mu$) with length $\sigma+u+1$ and the following additional properties: In the setting of Lemma 1.1 they must have a (u+1)-simple character $\vec{\kappa}^+_{simp}$; in the setting of Lemma 1.3 they must have a u-simple character $\vec{\kappa}_{simp}$ and a weak $(u+1)$-character $Weak(\vec{\kappa}^+_{simp})$.

Now, a few definitions that are only applicable when $\pi=1$.[68] *We denote by $\sum_{d\in D} a_d C^{d,i_1,\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)$ a generic linear combination of acceptable a-tensor fields with length $\sigma+u$, $a\geq\mu$, with simple character*

[68] See the notation above.

$pre\vec{\kappa}^+_{simp}$, and with an expression $\nabla^{(2)}_{ij}\phi_{u+1}\nabla^i\tilde{\phi}_{Min}$. If $a=\mu$ then we additionnaly require that if we formally replace the expression $\nabla^{(2)}_{ij}\phi_{u+1}\nabla^i\tilde{\phi}_{Min}$ by a factor $\nabla_j Y$ then the resulting tensor field is not forbidden in the sense of Lemma 4.6 in [6].

Finally, only in the setting of Lemma 1.1, we will denote by $\sum_{d\in D_{nc}} a_d C^{d,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots)$ a generic linear combination of tensor fields with length $\sigma+u$, a factor $\nabla^{(2)}_{ij}\phi_{u+1}\nabla^i\phi_{Min}$ ($_j$ is not a free index) and simple character $pre\vec{\kappa}^+_{simp}$, and with one of the free indices $_{i_1},\dots,_{\mu-1}$ being a derivative index. If this index belongs to a factor $\nabla^{(B)}\Omega_h$ tehn $B\geq 3$.

Armed with all the above notational conventions, we refer back to (2.8), and we set out to understand the form of the sublinear combinations:

$$CurvTrans[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

and $CurvTrans[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$.

Lemma 3.4

$$CurvTrans[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\sum_{d\in D^\sharp} a_d C^d_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})+\\ \sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})+\sum_{z\in Z} a_z C^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})\\ +\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u). \tag{3.32}$$

Lemma 3.5 *Consider any tensor field $C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$, $l\in L$ where none of the indices $_{k},_{l}$ in the crucial factor are free indices. Consider the special factor $S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ijkl}$ and denote by ρ the number of the indices $_{r_1},\dots,_{r_\nu},_j$ that are free in $C^{l,i_1\dots i_a}_g$, and we denote them by $_{i_1},\dots,_{i_\rho}$, for convenience. We claim that:*

$$
\begin{aligned}
&CurvTrans^I\{Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\}+\\
&CurvTrans^{II}\{Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\}=\\
&-\frac{1}{\nu+1}\sum_{y=1}^{\rho}Xdiv_{i_1}\dots X\hat{d}iv_{i_y}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_y}\phi_{u+1}\\
&+\sum_{p\in P}a_pXdiv_{i_1}\dots Xdiv_{i_a}C^{p,i_1,\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}\\
&+\sum_{j\in J}a_jC^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})\\
&+\sum_{d\in D}a_dXdiv_{i_1}\dots Xdiv_{i_a}C^{d,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\\
&+\sum_{d\in D^\sharp}a_dC^d_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+\sum_{z\in Z}a_zC^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).
\end{aligned}
\tag{3.33}
$$

Note: Let us observe that in the setting of Lemma 1.1, the tensor fields $C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_y}\phi_{u+1}$ are $(u+1)$-simply subsequent to $\vec{\kappa}^+_{simp}$. We will denote the sublinear combination:

$$
-\sum_{y=1}^{\rho}Xdiv_{i_1}\dots X\hat{d}iv_{i_y}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_y}\phi_{u+1}
$$

by $Leftover_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$.

Proof of the two Lemmas above: The proof follows straightforwardly by applying the transformation laws (2.1) and (3.1). □

Now, we focus on the case of the tensor fields $C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ where one of the indices ${}_{k},{}_{l}$ in the selected factor is a free index (with no loss of generality we assume that ${}_{k}$ is the free index ${}_{i_1}$ and ${}_{l}$ is not a free index). For convenience, we assume that the rest of the indices in $S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ijkl}$ that are free are precisely ${}_{i_2},\dots,{}_{i_{\rho+1}}$. For each such tensor field we will denote by ϵ the number of indices ${}_{r_1},\dots,{}_{r_\nu},{}_{j}$ in the crucial factor $S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ijkl}$ that are neither free nor contracting against a factor $\nabla\phi_{e_1},\dots\nabla\phi_{e_{\pi-1}}$.

Lemma 3.6 *With the notational conventions above we claim:*

$$CurvTrans^I\{Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\}+$$
$$CurvTrans^{II}\{Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\}=$$
$$-\frac{1}{\nu+1}\sum_{y=1}^{\rho}Xdiv_{i_1}\dots X\hat{div}_{i_{y+1}}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{y+1}}\phi_{u+1}$$
$$+\sum_{p\in P}a_pXdiv_{i_1}\dots Xdiv_{i_a}C^{p,i_1,\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}$$
$$-\frac{\epsilon}{\nu+1}Xdiv_{i_2}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+$$
$$\sum_{d\in D^\sharp}a_dC^d_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+$$
$$(\sum_{d\in D_{nc}}a_dXdiv_{i_1}\dots Xdiv_{i_{a-1}}C^{d,i_1,\dots,i_{a-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1}))+$$
$$\sum_{z\in Z}a_zC^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1});\tag{3.34}$$

here here the sublinear combination $\sum_{d\in D_{nc}}a_d\dots$ *arises only in the setting of Lemma 1.1, when* $Free(Max)>1$.

Definition 3.3 *We will denote the linear combination*

$$-\frac{1}{\nu+1}\sum_{y=1}^{\rho}Xdiv_{i_1}\dots X\hat{div}_{i_{y+1}}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{y+1}}\phi_{u+1}$$
$$-\frac{\epsilon}{\nu+1}Xdiv_{i_2}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\tag{3.35}$$

by

$$Leftover_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)].$$

Proof of Lemma 3.6: All the above claims follow by the definitions. Only for (3.34) we must also use the equation (3.1) also, for (3.34) we use the first Bianchi identity. □

In conclusion, we have shown that in the setting of Lemmas 1.1 and 1.3 (when the selected factor is of the from $S_*\nabla^{(\nu)}R_{ijkl}$), $CurvTrans[L_g]$ can now be expressed as:

$$\sum_{l\in L} a_l CurvTrans[Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1\dots,\Omega_p,\phi_1,\dots,\phi_u)]+$$
$$\sum_{j\in J} a_j CurvTrans[C^j_g(\Omega_1\dots,\Omega_p,\phi_1,\dots,\phi_u)] =$$
$$\sum_{l\in L} a_l Leftover_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1\dots,\Omega_p,\phi_1,\dots,\phi_u)]+$$
$$\sum_{d\in D^\sharp} a_d C^d_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u) + \sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)$$
$$+(\sum_{d\in D_{nc}} a_d Xdiv_{i_1}\dots Xdiv_{i_{a-1}} C^{d,i_1,\dots,i_{a-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1}))+$$
$$\sum_{d\in D} a_d Xdiv_{i_1}\dots Xdiv_{i_a} C^{d,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+$$
$$\sum_{p\in P} a_p Xdiv_{i_1}\dots Xdiv_{i_a} C^{p,i_1,\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+$$
$$\sum_{z\in Z} a_z C^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1}). \tag{3.36}$$

Our aim is now to "get rid" of all the sublinear combinations indexed in $D, D_{nc}, D^\sharp$, modulo introducing correction terms with $\sigma+u+1$ factors that are allowed in the conclusions of Lemma 1.1 and 1.3. The rest of this subsection is devoted to that goal.

In view of (2.8) and (3.36), we derive an equation:

$$0 = \big(\sum_{d\in D_{nc}} a_d Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} C^{d,i_1,\dots,i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\big)$$
$$+\sum_{d\in D} a_d Xdiv_{i_1}\dots Xdiv_{i_a} C^{d,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+$$
$$\sum_{d\in D^\sharp} a_d C^d_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u) + \sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u), \tag{3.37}$$

modulo complete contractions of length $\geq \sigma+u+1$. We then claim:

Lemma 3.7 *Refer to (3.37). We claim that we can write:*

$$
\begin{aligned}
&(\sum_{d\in D_{nc}} a_d Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} C^{d,i_1,\dots,i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u))+\\
&+\sum_{d\in D} a_d Xdiv_{i_1}\dots Xdiv_{i_a} C^{d,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+\\
&\sum_{d\in D^\sharp} a_d C^d_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\\
&=(\sum_{p\in P'} a_p Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} C^{p,i_1,\dots i_{\mu-1},i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1})+\\
&\sum_{a>\mu-1}\sum_{p\in P} a_p Xdiv_{i_1}\dots Xdiv_{i_a} C^{p,i_1,\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}\\
&+\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})+\sum_{z\in Z} a_z C^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).
\end{aligned}
\tag{3.38}
$$

Here the sublinear combination $\sum_{p\in P'}\dots$ arises only in the setting of Lemma 1.1. In that case, it stands for a generic linear combination of acceptable tensor fields with a $(u+1)$-simple character $\vec{\kappa}^+_{simp}$ and with fewer than $Free(Max)$ free indices in the selected (crucial) factor. Equation (3.38) holds modulo terms of length $\geq \sigma+u+2$.

Proof of Lemma 3.7:

We show the above via an induction. However the base case of our induction depends on which setting we are in. In the setting of Lemma 1.3 the linear combination $\sum_{d\in D_{nc}}\dots$ is not present. Also, in the case of Lemma 1.1, if $Free(Max)=1$ then $\sum_{d\in D_{nc}}\dots$ is not present. In those cases we may skip to after equation (3.42). In the setting of Lemma 1.1 with $Free(Max)\geq 2$ we must first "get rid" of the sublinear combination $\sum_{d\in D_{nc}}\dots$.

So, we now assume that $D_{nc}\neq\emptyset$. We refer to (3.37), and we recall that all the tensor fields involved have a *fixed* simple character, which we have denoted by $pre\vec{\kappa}^+_{simp}$. Picking out the sublinear combination in (3.37) which consists of complete contractions with a factor $\nabla^{(2)}\phi_{u+1}$ we derive a new equation:

$$
\begin{aligned}
&\sum_{d\in D_{nc}} a_d X_*div_{i_1}\dots X_*div_{i_{\mu-1}} C^{d,i_1,\dots,i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+\\
&+\sum_{d\in D} a_d X_*div_{i_1}\dots X_*div_{i_a} C^{d,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+\\
&\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)=0,
\end{aligned}
\tag{3.39}
$$

which holds modulo complete contractions of length $\geq\sigma+u+1$. Here X_*div_i stands for the sublinear combination in $Xdiv_i$ where ∇_i is not allowed to hit

the factor $\nabla^{(2)}\phi_{u+1}$. Therefore, applying the eraser to the factor $\nabla\phi_{Min}$, in the above equation and then the Lemma 4.10 from [6] (which we are now inductively assuming because we have lowered the absolute value of the weight)[69] we derive that there is a linear combination of acceptable μ-tensor fields, with a simple character $pre\vec{\kappa}^+_{simp}$, say

$$\sum_{h\in H} a_h C^{h,i_1\dots,i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u),$$

each with a factor $\nabla^{(2)}_{ij}\phi_{u+1}\nabla^i\tilde{\phi}_{Min}$ so that:

$$\begin{aligned}
&\sum_{d\in D_{nc}} a_d C^{d,i_1,\dots,i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon \\
&- X_*div_{i_\mu}\sum_{h\in H} a_h C^{h,i_1\dots,i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon \\
&= \sum_{j\in J} a_j C^{j,i_1,\dots,i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon,
\end{aligned} \tag{3.40}$$

where each $C^{j,i_1,\dots,i_{\mu-1}}_g$ is simply subsequent to $pre\vec{\kappa}^+_{simp}$ (the above holds modulo terms of length $\geq \sigma+u+2$).

Now, two observations: Firstly, in the generic notation we have introduced, we have:

$$\begin{aligned}
&\sum_{h\in H} a_h C^{h,i_1\dots,i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u) \\
&= \sum_{d\in D} a_d C^{d,i_1\dots,i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u).
\end{aligned} \tag{3.41}$$

Secondly, since (3.40) holds formally, by making the $\nabla\upsilon$'s into $Xdiv$'s (using the last Lemma in the Appendix of [3]), we derive:

[69] By the definition of the terms $\sum_{d\in D_{nc}}\dots$ there is no danger of falling under a "forbidden case" of that Lemma.

$$
\begin{aligned}
&\sum_{d\in D_{nc}} a_d Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} C^{d,i_1,\dots,i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\\
&- Xdiv_{i_1}\dots Xdiv_{i_\mu}\sum_{h\in H} a_h C^{h,i_1\dots,i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)=\\
&\sum_{d\in D^\sharp} a_d C^d_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+\\
&\sum_{p\in P'} a_p Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} C^{p,i_1,\dots,i_{\mu-1},i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+\sum_{z\in Z} a_z C^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),
\end{aligned}
\tag{3.42}
$$

(modulo length $\geq \sigma+u+2$). Thus, by virtue of the two above equations we are reduced to showing our claim under the extra assumption that $D_{nc}=\emptyset$.

Next, we refer back to (3.37) and we claim that we can write:

$$
\begin{aligned}
&\sum_{d\in D} a_d Xdiv_{i_1}\dots Xdiv_{i_a} C^{d,i_1,\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)=\\
&\sum_{d\in D'} a_d Xdiv_{i_1}\dots Xdiv_{i_a} C^{d,i_1,\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+\\
&\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+\sum_{z\in Z} a_z C^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),
\end{aligned}
\tag{3.43}
$$

modulo length $\geq \sigma+u+2$.

Here
$$\sum_{d\in D'} a_d C^{d,i_1,\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)$$
stands for a generic linear combination of tensor fields in the general form (2.15) with a factor $\nabla^{(m)}\phi_{u+1}$ that contracts according to the pattern $\nabla^{(3)}_{sr_1z}\phi_{u+1}\nabla^{r_1}\phi_{Min}$, where neither of the indices ${}_s,{}_z$ is free.

The above equation can be proven as follows: Firstly, apply the eraser (in the equation (3.37)) to the factor $\nabla\phi_{Min}$ that is contracting against the factor $\nabla^{(2)}\phi_{u+1}$ (and thus obtain a new true equation which we denote by (3.37)'), and then pick out the sublinear combination that contains a factor $\nabla\phi_{u+1}$ with only one derivative (and thus obtain another true equation which we denote by (3.37)")–for the next construction we re-name the function ϕ_{u+1} Y.[70] We are then in a position to apply Corollary 2 from [6] (if $\sigma>3$) or Lemma 4.7 from

[70](This is after we have applied the eraser to the factor $\nabla\phi_{Min}$ that contracted against $\nabla^{(B)}\phi_{u+1}$).

[6] (if $\sigma = 3$) to the new true equation (3.37)", and derive (3.43): We start with the case $\sigma = 3$: Observe that (3.37)" satisfies the requirements of Lemma 4.7 by weight considerations since we are assuming that the assumption of Lemma 1.1 does not contain "forbidden" tensor fields. Thus, we apply Lemma 4.7 in [6] and in the end we replace the function Y by a function $\nabla_s\phi_{u+1}\nabla^s\phi_{Min}$. Then, picking out the sublinear combination with the function $\nabla\phi_{Min}$ differentiated only once,[71] we derive (3.43). Now, the case $\sigma > 3$ follows by the exact same argument, only instead of Lemma 4.7 we apply Corollary 2 from [6]. Corollary 2 can be applied to (3.37)" since by definition there is no danger of falling under a "forbidden case" of that Lemma.

But then, we refer to (3.43) and we observe that we can write:

$$\begin{aligned}
&\sum_{d\in D'} a_d Xdiv_{i_1}\dots Xdiv_{i_a} C^{d,i_1,\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u) = \\
&\sum_{d\in D^\sharp} a_d C^d_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u) + \\
&\sum_{p\in P} a_p Xdiv_{i_1}\dots Xdiv_{i_a} C^{d,i_1,\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1} + \\
&\sum_{z\in Z} a_z C^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).
\end{aligned} \tag{3.44}$$

This just follows from the identity:

$$(\nabla_a\nabla^{(2)}_{r_1r_2}\phi_{u+1})\nabla^{r_1}\phi_{Min} = \nabla^{(3)}_{r_1ar_2}\phi_{u+1}\nabla^{r_1}\phi_{Min} + R_{ar_1kr_2}\nabla^k\phi_{u+1}. \tag{3.45}$$

Thus, replacing (3.43) and (3.44) into (3.37) we are now reduced to showing Lemma 3.7 in the case $D_{nc} = \emptyset, D = \emptyset$.

Now, one more Lemma:

Lemma 3.8 *Assume (3.37) with $D_{nc} = D = \emptyset$. We claim:*

$$\begin{aligned}
&\sum_{d\in D^\sharp} a_d C^d_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u) = \sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1}) \\
&+ \sum_{z\in Z} a_z C^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),
\end{aligned} \tag{3.46}$$

(modulo length $\geq \sigma + u + 2$), where here each C^j on the right hand side has length $\sigma + u + 1$ and a weak character $Weak(\vec{\kappa}^+_{simp})$ but also has the factor $\nabla\phi_{Min}$ contracting against a derivative index of the factor $\nabla^{(m)}R_{ijkl}$ (and thus is simply subsequent to $\vec{\kappa}^+_{simp}$).

[71] Observe that this sublinear combination will vanish separately.

Moreover, we claim that we can write:

$$\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u) =$$
$$\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1}) + \sum_{z\in Z} a_z C^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}), \tag{3.47}$$

where each C^j_g on the right hand side is simply subsequent to $\vec{\kappa}^+_{simp}$.

Proof: In order to introduce a strict dichotomy between the linear combinations indexed in $D^\sharp, J$, we permute the indices in the factor $\nabla^{(B)}\phi_{u+1}$ in each $C^j_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)$ to make the factors $\nabla\tilde{\phi}_{Min},\nabla\phi'_{e_1},\dots\nabla\phi'_{e_{\pi-1}}$ contract against the first π indices, in that order. We can clearly do this modulo introducing correction terms in the general form:

$$\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1}) + \sum_{z\in Z} a_z C^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).$$

Then, we inquire on the number of derivatives on the factor $\nabla^{(B)}\phi_{u+1}$ in each C^j. If $B>\pi+1$ we just re-name C^j into C^d and index it in $D^\sharp$. We are reduced to showing our claim under the hypothesis that all C^j have $B\le\pi+1$ derivatives on the factor $\nabla^{(B)}\phi_{u+1}$. We proceed under that assumption.

Trivially, since (3.37) holds formally and since we are assuming $D_{nc}=D=\emptyset$, we derive that:

$$\sum_{d\in D^\sharp} a_d lin C^d_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u) = 0,$$

$$\sum_{j\in J} a_j lin C^j_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u) = 0.$$

Therefore our claim (3.47) follows by just repeating the permutations by which we make the left hand sides of the last two equations formally zero, whereas (3.46) follows by the same fact, and also by using the fact that the first π indices in the factor $\nabla^{(B)}\phi_{u+1}$ are not permuted (which can be proven as usual using the eraser). $\Box$

Thus, in the setting of Lemma 1.1 and of Lemma1.3 (if the selected factor is in the form $S_*\nabla^{(\nu)}R_{ijkl}$) we have shown that:

$$
\begin{aligned}
&\sum_{l\in L} a_l CurvTrans[Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&\sum_{j\in J} a_j CurvTrans[C^j_g(\Omega_1\dots,\Omega_p,\phi_1,\dots,\phi_u)] =\\
&\sum_{l\in L} a_l Leftover_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&(\sum_{p\in P'} a_p Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} C^{p,i_1,\dots i_{\mu-1},i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1})+\\
&\sum_{p\in P} a_p Xdiv_{i_1}\dots Xdiv_{i_a} C^{p,i_1,\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})+\sum_{z\in Z} a_z C^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1});
\end{aligned}
\tag{3.48}
$$

(the linear combination $\sum_{p\in P'}\dots$ arises only in the setting of Lemma 1.1).

4 A study of the sublinear combinations $LC[L_g]$ and $W[L_g]$ in (6.1). Computations and cancellations.

4.1 General discussion of ideas:

The main conclusions we retain from the previous two subsections are equations (3.48) and (3.29), (3.30), (3.31).

We will denote by $CurvTrans^{study}[L_g]+\sum_{z\in Z}\dots$ the right hand sides of those equations.[72] Thus, we have shown that the sublinear combinations $CurvTrans[L_g]$ can be *re-expressed* as linear combinations $CurvTrans^{study}[L_g]+\sum_{z\in Z}\dots$.

We then *replace* $CurvTrans[L_g]$ by $CurvTrans^{study}[L_g]+\sum_{z\in Z}\dots$ in (6.1), obtaining a new equation:

$$
CurvTrans^{study}[L_g]+LC[L_g]+W[L_g]+\sum_{z\in Z}\dots=0,
\tag{4.1}
$$

which holds modulo complete contractions of length $\geq \sigma+u+2$ (notice all complete contractions in the LHS of the above have length $\sigma+u+1$). Thus, picking out the sublinear combination of complete contractions with length $\sigma+$

[72] $\sum_{z\in Z}\dots$ stands for the sublinear combination of complete contractions indexed in Z in the right hand sides of (3.48) and (3.29), (3.30), (3.31), and $CurvTrans^{study}[L_g]$ stands for the terms with a factor $\nabla\phi_{u+1}$ in the RHSs in (3.48) and (3.29), (3.30), (3.31).

$u+1$ and with a factor $\nabla\phi_{u+1}$ (with only one derivative)[73] we obtain a new equation:

$$CurvTrans^{study}[L_g]+LC[L_g]+W[L_g]=0, \tag{4.2}$$

which holds modulo complete contractions of length $\geq \sigma+u+2$.

For the rest of this paper and the next one in this series, we will try to understand the sublinear combinations $LC_{\phi_{u+1}}[L_g]+W_{\phi_{u+1}}[L_g]$ in the above equation.

We now focus on the sublinear combinations $LC[L_g]$ and $W[L_g]$ in (6.2) which by definition consist of terms with length $\sigma+u+1$ *and* have a weak $(u+1)$-character $Weak(\vec{\kappa}^+_{simp})$. We recall that $LC[L_g]$ and $W[L_g]$ were defined to be specific sublinear combinations of $Image^{1,+}_{\phi_{u+1}}[L_g]$: $LC[L_g]$ stands for the sublinear combination that arises *in* $Image^{1,+}_{\phi_{u+1}}[L_g]$ by applying the formula (2.2), and $W[L_g]$ stands for the sublinear combination of terms that arises by virtue of the transformation $R_{ijkl}(e^{2\phi_{u+1}}g)\rightarrow e^{2\phi_{u+1}}R_{ijkl}(g)$.

Furthermore, recall that we have broken up

$$LC[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_{u+1})]$$

into two sublinear combinations:

$$LC_\Phi[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_{u+1})],$$

$$LC^{No\Phi}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_{u+1})].$$

Recall that $LC_\Phi[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_{u+1})]$ is the sublinear combination in $LC[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ that arises by applying (2.2) to two indices, at least one of which is contracting against a factor $\nabla\phi_y$. Recall (2.17).

Recall that $LC^{No\Phi}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ stands for the sublinear combination that arises in $LC[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ by applying (2.2) to two indices that are not contracting against a factor $\nabla\phi_y$. We have analogously defined $LC^{No\Phi}[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$.

Our aim for this section is to understand the sublinear combinations:

$$\begin{aligned}&LC^{No\Phi}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\&Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1\cdot\phi_{u+1},\dots,\Omega_p,\phi_1,\dots,\phi_u)+\dots+\\&Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_X\cdot\phi_{u+1},\dots,\Omega_p,\phi_1,\dots,\phi_u),\end{aligned} \tag{4.3}$$

$$\begin{aligned}&LC^{No\Phi}[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+C^j_g(\Omega_1\cdot\phi_{u+1},\dots,\Omega_p,\phi_1,\dots,\phi_u)\\&+\dots+C^j_g(\Omega_1,\dots,\Omega_X\cdot\phi_{u+1},\dots,\Omega_p,\phi_1,\dots,\phi_u),\end{aligned} \tag{4.4}$$

[73]This sublinear combination must vanish separately.

$$W[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)], \tag{4.5}$$

$$W[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]. \tag{4.6}$$

(Recall that $\Omega_1,\dots,\Omega_X$ are the factors Ω_h that are not contracting against a factor $\nabla\phi_h$ in $\vec{\kappa}_{simp}$).

A few notes: In the setting of Lemmas 1.1 and 1.2, we will be able to *discard* sublinear combinations in the above four linear combinations that consist of complete contractions with length $\sigma+u+1$ and where the factor $\nabla\phi_{u+1}$ is contracting against a derivative index in the crucial factor when it is in the form $\nabla^{(m)}R_{ijkl}$,[74] or contracting against any index ${}_{r_1},\dots,{}_{r_\nu},{}_j$ in the crucial factor if it is in the form $S_*\nabla^{(\nu)}R_{ijkl}$.[75] We denote generic such linear combinations by $\sum_{q\in Q}a_qC^q(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$. The contractions C^q_g do not have to be acceptable. In particular, they might have a factor $\nabla\Omega_h$. Observe that such generic linear combinations are *allowed* in the right hand sides of Lemmas 1.1 and 1.2: They are special cases of the linear combinations $\sum_{t\in T}\dots$ in the right hand sides of those Lemmas.

We introduce another notational convention we will be using throughout this section:

Definition 4.1 *We denote by* $\sum_{h\in H}a_hC^{i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}$ *a generic linear combination of the forms* $\sum_{p\in P}a_p\dots$ *as in the statements of Lemmas 1.1 and 1.2 (if we are in the setting of those Lemmas), or a generic linear combination of the form* $\sum_{t\in T_1\bigcup T_2\bigcup T_3\bigcup T_4}\dots$*, in the notation of Lemma 1.3 (if we are in the setting of case A of that Lemma).*

We will be calling the tensor fields in those linear combinations "contributors".

Finally, we also recall that $\sum_{j\in J}a_jC^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$ stands for a generic linear combination of complete contractions of length $\sigma+u+1$, with a weak $(u+1)$-character $Weak(\vec{\kappa}^+_{simp})$ and which are u-simply subsequent to $\vec{\kappa}_{simp}$).

Now, we proceed to study the four expressions (4.3), (4.4), (4.5), (4.6).

The easiest to study are (4.4) and (4.6). We straightforwardly observe that for each $j\in J$ we must have:

$$\begin{aligned}&LC^{No\Phi}[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+C^j_g(\Omega_1\cdot\phi_{u+1},\dots,\Omega_p,\phi_1,\dots,\phi_u)\\&+\dots+C^j_g(\Omega_1,\dots,\Omega_X\cdot\phi_{u+1},\dots,\Omega_p,\phi_1,\dots,\phi_u)\\&=\sum_{j\in J}a_jC^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),\end{aligned} \tag{4.7}$$

[74] i.e. in the setting of Lemma 1.2

[75] i.e. if we are in the setting of Lemma 1.1.

$$W[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\sum_{j\in J}a_jC^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}). \quad (4.8)$$

The harder challenge is to understand the linear combinations (4.3), (4.5). In order to understand these two sublinear combinations, we will break them up into further sublinear combinations:

Definition 4.2 *We define* $LC^{No\Phi}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ *to stand for the sublinear combination that arises in*

$$Image^1_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

when we apply the transformation law (2.2) to any two indices in the tensor field $C^{l,i_1\dots i_a}_g$ *(we will call these original indices), that are not contracting against a factor* $\nabla\phi_y, y\le u$ *and bring out a factor* $\nabla_{i_*}\phi_{u+1}$ *for which* $_{i_*}$ *is either contracting against the crucial factor or is a free index.*

We also define

$$LC^{No\Phi,div}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

to stand for the sublinear combination that arises in

$$Image^1_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

when we apply (2.2) to two indices in the same factor, and at least one of those indices is of the form ∇^{i_h} $(1\le h\le a)$ *(we call such indices divergence indices), and the other is not contracting against a factor* $\nabla\phi_y$.

Secondly, we denote by $W[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ *the sublinear combination in*

$$Image^1_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] \quad (4.9)$$

that arises when we replace a factor $\nabla^{(m)}_{r_1\dots r_m}R_{ijkl}$ *by a factor* $\nabla^{(m)}_{r_1\dots r_m}[e^{2\phi_{u+1}}R_{ijkl}]$ *(by virtue of (2.1)) and then bring out an expression* $e^{2\phi_{u+1}}\nabla^{(m-1)}R_{ijkl}\nabla\phi_{u+1}$ *by hitting the factor* $e^{2\phi_{u+1}}$ *by one of the derivatives* $\nabla_{r_1},\dots,\nabla_{r_m}$.

Furthermore, we define $W^{div}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ *to stand for the sublinear combination in* $W[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ *that arises by picking a factor* $\nabla^{i_y\dots i_z}\nabla^{(m)}_{r_1\dots r_m}R_{ijkl}$ *in some summand in* $Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ *(*$^{i_y},\dots,^{i_z}$ *are divergence indices) and replacing it by* $\nabla^{i_y\dots i_z}[e^{2\phi_{u+1}}\nabla^{(m)}_{r_1\dots r_m}R_{ijkl}]$ *and then bringing out an expression* $e^{2\phi_{u+1}}\nabla^{(m'-1)}R_{ijkl}\nabla\phi_{u+1}$ *by hitting the factor* $e^{2\phi_{u+1}}$ *by one of the divergence indices* $\nabla^{i_y},\dots,\nabla^{i_z}$.

It then follows directly from the above definition that:

$$
\begin{aligned}
&LC^{No\Phi}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\\
&+W[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&LC^{No\Phi,div}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&W^{div}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&Xdiv_{i_1}\dots Xdiv_{i_a}W[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)],
\end{aligned}
\tag{4.10}
$$

subject to a small clarification regarding the notion of $Xdiv$ in the linear combinations $Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$:

Definition 4.3 *We have defined $Xdiv_{i_y}$ to stand for the sublinear combination in div_{i_y} where ∇_{i_y} is not allowed to hit the factor to which ${}_{i_y}$ belongs, nor any factor $\nabla\phi_h$.*

Now, for each free index ${}_{i_y}$ (that belongs to a factor T in the form $\nabla^{(p)}\Omega_h$ or $\nabla^{(m)}R_{ijkl}$), and each tensor field $C^{*,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$ in $LC^{No\Phi}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$, either ${}_{i_y}$ still belongs to the factor $\nabla^{(p)}\Omega_h$ or $\nabla^{(m)}R_{ijkl}$ in $C^{*,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$, or ${}_{i_y}$ belongs to an un-contracted metric tensor g, or ${}_{i_y}$ belongs to a factor $\nabla_{i_y}\phi_{u+1}$. In the first case, we define $Xdiv_{i_y}C^{*,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$ as in the above paragraph. In the second case, we see that the un-contracted metric tensor in $C^{*,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$ must have arisen by applying the third summand on the right hand side on (2.2) to a pair of indices $(\nabla_a,{}_b)$ in a factor T of the form $\nabla^{(p)}\Omega_h$ or $\nabla^{(m)}R_{ijkl}$, where either ∇_a or ${}_b$ is the free index ${}_{i_y}$. In that case, we define $Xdiv_{i_y}C^{*,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$ to stand for the sublinear combination in $div_{i_y}C^{*,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$ where ∇_{i_y} is not allowed to hit the factor T nor the uncontracted metric tensor nor the factor $\nabla_{i_y}\phi_{u+1}$ nor any $\nabla\phi_h, h\le u$. In the third case, the expression $\nabla_{i_y}\phi$ must have arisen by applying one of the first two terms in the transformation law (2.2) to the factor T. In that case, we define $Xdiv_{i_y}$ to stand for the sublinear combination in div_{i_y} where ∇_{i_y} is not allowed to hit the factor T nor the factor $\nabla_{i_y}\phi_{u+1}$ nor any $\nabla\phi_h, h\le u$.

With this clarification equation (4.10) just follows by the definition of $LC^{No\Phi}_{\phi_{u+1}}[\dots]$ and $W[\dots]$ and the transformation law (2.2). Now, we will subdivide the right hand side of (4.10) into further sublinear combinations:

A study of the right hand side of (4.10):

We will firstly study the last two lines in (4.10).
We define

$$W^{targ}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

to stand for the sublinear combination in

$$W[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

for which $\nabla\phi_{u+1}$ is contracting against the crucial factor.

We define

$$W^{free}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

to stand for the sublinear combination in $W[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ for which the index ${}_\alpha$ in $\nabla_\alpha\phi_{u+1}$ is a free index.

We also define

$$W^{targ,div}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

to stand for the sublinear combination in $W^{div}[\dots]$ where in addition the factor $\nabla\phi_{u+1}$ that we bring out is contracting against the (a) crucial factor.

Thus it follows that:

$$\begin{aligned}
&W[\sum_{l\in L}a_l Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&W^{targ,div}[\sum_{l\in L}a_l Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&\sum_{l\in L}a_l Xdiv_{i_1}\dots Xdiv_{i_a}W^{targ}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&\sum_{l\in L}a_l Xdiv_{i_1}\dots Xdiv_{i_a}W^{free}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)].
\end{aligned}\tag{4.11}$$

In view of the above, we study the three sublinear combinations in the right hand side of (4.11) separately. We derive by definition:

$$\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}W^{free}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&\sum_{q\in Q}a_q C^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).
\end{aligned}\tag{4.12}$$

Next, we will study the remaining two sublinear combinations in the right hand side of (4.11) together: Recall that the total number of factors $\nabla^{(m)}R_{ijkl}$ or $S_*\nabla^{(\nu)}R_{ijkl}$ is $s=\sigma_1+\sigma_2$. Now, in the setting of Lemma 1.1, if $l\in L_\mu$ and $C^{l,i_1\dots i_a}_g$ has one free index (say ${}_{i_1}$ with no loss of generality) being the index ${}_k$ in the crucial factor, we have:

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_\mu}W^{targ}[C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&W^{targ,div}[Xdiv_{i_1}\dots Xdiv_{i_\mu}C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&2(s-1)Xdiv_{i_2}\dots Xdiv_{i_\mu}C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\
&\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_\mu}C^{h,i_1\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}\\
&+\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),
\end{aligned}
\tag{4.13}
$$

where

$$\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_\mu}C^{h,i_1\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}$$

stands for a generic linear combination of *acceptable* contributors.[76] If the tensor field $C^{l,i_1\dots i_a}_g$ does not have a free index ${}_{i_h}$ that is an index ${}_k$ or ${}_l$ in the crucial factor, then we calculate:

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_\mu}W^{targ}[C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&W^{targ,div}[Xdiv_{i_1}\dots Xdiv_{i_\mu}C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_\mu}C^{h,i_1\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}\\
&+\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).
\end{aligned}
\tag{4.14}
$$

If $l\in L\setminus L_\mu$, (hence $a>\mu$):

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}W^{targ}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&W^{targ,div}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_{a-1}}C^{h,i_1\dots i_{a-1},i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}\\
&+\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).
\end{aligned}
\tag{4.15}
$$

On the other hand, in the setting of Lemma 1.2, recall that $I_{l,*}$ stands for the index set of special free indices in factors $\nabla^{(m)}R_{ijkl}$ in $C^{l,i_1\dots i_\mu}_g$; we then compute:

[76]See the definition 4.1.

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_\mu}W^{targ}[C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&W^{targ,div}[Xdiv_{i_1}\dots Xdiv_{i_\mu}C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&2(s-1)\sum_{i_h\in I_{*,l}}Xdiv_{i_1}\dots X\hat{d}iv_{i_h}\dots Xdiv_{i_\mu}C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\\
&\nabla_{i_1}\phi_{u+1}+\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_\mu}C^{h,i_1\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}\\
&+\sum_{z\in Z}a_zC^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})+\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).
\end{aligned}
\tag{4.16}
$$

Moreover, in the setting of Lemma 1.2 we again have the equation (4.15).

A study of $LC^{No\Phi,div}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$, $Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ *in the RHS of (4.10):*

Next, we will study the first two lines in the right hand side of (4.10).
We define

$$LC^{No\Phi,targ}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

to stand for the sublinear combination in

$$LC^{No\Phi}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

for which $\nabla\phi_{u+1}$ is contracting against the (a) crucial factor.

Furthermore, we define

$$LC^{No\Phi,free}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

to stand for the sublinear combination in

$$LC^{No\Phi}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

for which the index in $\nabla\phi_{u+1}$ is a free index. It follows (by just applying the notational conventions of definition 4.2) that:

$$
\begin{aligned}
&LC^{No\Phi}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=LC^{No\Phi,targ}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\\
&+LC^{No\Phi,free}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)].
\end{aligned}
\tag{4.17}
$$

We easily observe that:

$$Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,free}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}). \tag{4.18}$$

These above facts were straightforward. We now explain a more delicate fact. We consider the set $I_l=\{_{i_1},\dots,_{i_a}\}$ of free indices. We break it up into two subsets. We say $_i\in I_1\subset I$ if $_i$ belongs to the crucial factor. We say $_i\in I_2$ if it does not belong to the crucial factor. We then denote by

$$LC^{No\Phi,div,I_2}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

the sublinear combination in

$$LC^{No\Phi,div}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

that arises when we apply the transformation law (2.2) to a pair of indices $(\nabla_{i_y},_a)$ where $_{i_y}\in I_2$ and $_a$ is an index in the tensor field

$$C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$$

, *or* $_a$ is another derivative index $\nabla_{i_x},_{i_x}\in I_2$ *or* $_a$ is a derivative index ∇_{i_x} with $_{i_x}\in I_1$. We also denote by

$$LC^{No\Phi,div,I_1}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

the sublinear combination in

$$LC^{No\Phi,div}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

that arises when we apply the transformation law (2.2) to a pair of indices $(\nabla_{i_y},_b)$ where $_{i_y}\in I_1$ and $_b$ is an index in the tensor field $C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ (same as before) *or* $_b$ is another derivative index $\nabla_{i_x},\ _{i_x}\in I_1$.

We derive by definition that:

$$\begin{aligned}
&LC^{No\phi}_{\phi_{u+1}}[\sum_{l\in L}a_lXdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&LC^{No\phi,div,I_1}_{\phi_{u+1}}[\sum_{l\in L}a_lXdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&LC^{No\phi,div,I_2}_{\phi_{u+1}}[\sum_{l\in L}a_lXdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&\sum_{l\in L}a_lXdiv_{i_1}\dots Xdiv_{i_a}LC^{No\phi,targ}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&\sum_{l\in L}a_lXdiv_{i_1}\dots Xdiv_{i_a}LC^{No\phi,free}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)].
\end{aligned} \tag{4.19}$$

We then observe that:

$$
\begin{aligned}
&LC^{No\Phi,div,I_2}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).
\end{aligned}
\tag{4.20}
$$

Now, let us denote by $X_*div_{i_1}\dots X_*div_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1\cdot\phi_{u+1},\dots,\Omega_p,\phi_1,\dots,\phi_u)$, $\dots$, $X_*div_{i_1}\dots X_*div_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_X\cdot\phi_{u+1},\dots,\Omega_p,\phi_1,\dots,\phi_u)$ the sublinear combination in $Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1\cdot\phi_{u+1},\dots,\Omega_p,\phi_1,\dots,\phi_u)$, $\dots$, $Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_X\cdot\phi_{u+1},\dots,\Omega_p,\phi_1,\dots,\phi_u)$ that arises when each $\nabla_i, i\in I_1$ is not allowed to hit the factor ϕ_{u+1}.

By definition, it follows that for each $h, 1\le h\le X$:

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_h\cdot\phi_{u+1},\dots,\Omega_p,\phi_1,\dots,\phi_u)=\\
&\sum_{i_y\in I_1}Xdiv_{i_1}\dots X\hat{div}_{i_y}\dots Xdiv_{i_\mu}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_y}\phi_{u+1}+\\
&X_*div_{i_1}\dots X_*div_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_h\cdot\phi_{u+1},\dots\Omega_p,\phi_1,\dots,\phi_u).
\end{aligned}
\tag{4.21}
$$

Definition 4.4 *We consider each linear combination*

$$LC^{No\Phi,targ}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

and we break it into two sublinear combinations:

$$LC^{No\Phi,targ,A}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

will stand for the sublinear combination that arises when we apply the transformation law (2.2) to any factor other than the selected one.

$$LC^{No\Phi,targ,B}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

will stand for the sublinear combination that arises when we apply the transformation law (2.2) to the (a) selected factor.

We then compute another delicate cancellation:

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,A}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&X_*div_{i_1}\dots X_*div_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1\cdot\phi_{u+1},\dots,\Omega_p,\phi_1,\dots,\phi_u)+\dots\\
&+X_*div_{i_1}\dots X_*div_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_X\cdot\phi_{u+1},\dots,\Omega_p,\phi_1,\dots,\phi_u)=\\
&\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_a}C^{h,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})+\sum_{z\in Z}a_zC^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1});
\end{aligned}
\tag{4.22}
$$

here each $C_g^{h,i_1\dots i_a,i_*}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}$ is an *acceptable* contributor.[77]

Next, we seek to understand the hardest linear combination in (4.19):

$$\begin{aligned}&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\&LC^{No\Phi,div,I_1}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)],\end{aligned}\tag{4.23}$$

where we recall that we have denoted by

$$LC^{No\Phi,div,I_1}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

the sublinear combination in

$$LC^{No\Phi,div}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

that arises when we apply the transformation law (2.2) to a pair of indices $(\nabla_{i_y},{}_a)$ where ${}_{i_y}\in I_1$ and ${}_a$ is *not* a divergence index $\nabla_{i_y}, i_y\in I_2$ (we have already counted those pairs).

Firstly, we present our claim in the setting of Lemma 1.1. Recall that in this setting the selected(=crucial) factor is unique.

We introduce some language conventions in order to formulate our claim:

Definition 4.5 *We consider each tensor field $C^{l,i_1\dots i_a}_g, l\in L$, and we denote by γ_l the total number of indices (free and non-free) that do not belong to the crucial factor and are not contracting against a factor $\nabla\phi_y$. We also recall that the number of free indices that belong to the crucial factor is $|I_1|$, and we let ν_l stand for the number of derivatives on the crucial factor $S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ijkl}$. We denote by ϵ_l the number of indices in the form ${}_{r_1},\dots,{}_{r_\nu},{}_j$ in the crucial factor that are not free and are not contracting against a factor $\nabla\phi_y$.*

Note: By abuse of notation, we will write γ,ν,ϵ instead of $\gamma_l,\nu_l,\epsilon_l$.

Lemma 4.1 *Consider the setting of Lemma 1.1, and consider any tensor field $C^{l,i_1\dots i_a}_g$, $l\in L$,[78] which has a special free index ${}_{i_1}={}_k$ in the crucial factor. Then:*

[77] See definition 4.1.

[78] Recall that by hypothesis $a\geq\mu$.

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&LC^{No\Phi,div,I_1}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&-(\gamma+\frac{\nu\cdot\epsilon}{\nu+1}-1)Xdiv_{i_2}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\
&\sum_{t\in T}a_tXdiv_{i_1}\dots Xdiv_{i_{a-1}}C^{t,i_1\dots i_{a-1},i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_\mu}C^{h,i_1\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1});
\end{aligned}
\tag{4.24}
$$

here

$$\sum_{t\in T}a_tC^{t,i_1\dots i_{a-1},i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}$$

stands for a generic linear combination of acceptable $(a-1)$-tensor fields for which ${}_{i_*}$ is the index ${}_k$ in the crucial factor, but we have fewer than $|I_1|-1$ free indices in the crucial factor. In particular, if $a=\mu$, then the $(\mu-1)$-tensor field will have a refined double character that is doubly subsequent to each $\vec{L^z}'$, $z\in Z'_{Max}$.

If $C^{l,i_1\dots i_a}_g$ does not have a free index in the position ${}_k$ or ${}_l$ in the selected factor $S_*\nabla^{(\nu)}R_{ijkl}$, then:

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&LC^{No\Phi,div,I_1}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_a}C^{h,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).
\end{aligned}
\tag{4.25}
$$

Proof of Lemma 4.1:

We start with the first claim, which is the hardest. We will show the above by breaking the left hand side into numerous sublinear combinations. Recall that we are assuming that the free index ${}_{i_1}$ is the index ${}_k$ in the crucial factor $S_*\nabla^{(\nu)}R_{ijkl}$, while the other free indices that belong to the crucial factor are ${}_{i_2},\dots,{}_{i_{|I_1|}}$. Firstly, let us analyze the sublinear combination

$$LC^{No\Phi,div,I_1}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)].$$

We break this sum into four sublinear combinations: First, we consider the sublinear combination that arises when we apply the transformation law (2.2) to a

pair of indices $(\nabla_{i_1}, {}_b)$ where ${}_b$ is an original index in $C^{l,i_1\dots i_a}_g$. Secondly, we consider the sublinear combination that arises when we apply the transformation law (2.2) to a pair of divergence indices, $(\nabla_{i_1}, \nabla_{i_k})$, $2 \le k \le |I_1|$. Thirdly, we consider the sublinear combination that arises when we apply the transformation law (2.2) to a pair of divergence indices $(\nabla_{i_k}, \nabla_{i_l})$, $2 \le k, l \le |I_1|$. Fourthly, we consider the sublinear combination that arises when we apply the transformation law (2.2) to a pair of indices $({}_{i_k}, {}_b)$, $2 \le k \le |I_1|$ and ${}_b$ being an original index in $C^{l,i_1\dots i_a}_g$. We respectively denote those sublinear combinations by $LC^{No\Phi,div,I_1,\alpha}_{\phi_{u+1}}[\dots]$, $LC^{No\Phi,div,I_1,\beta}_{\phi_{u+1}}[\dots]$, $LC^{No\Phi,div,I_1,\gamma}_{\phi_{u+1}}[\dots]$, $LC^{No\Phi,div,I_1,\delta}_{\phi_{u+1}}[\dots]$.

We then observe that:

$$\begin{aligned}&LC^{No\Phi,div,I_1,\alpha}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] =\\ &-(\gamma-1)C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\\ &+\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).\end{aligned} \tag{4.26}$$

The second sublinear combination is a little more complicated.

$$\begin{aligned}&LC^{No\Phi,div,I_1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] =\\ &-(|I_1|-1)Xdiv_{i_2}\dots Xdiv_{i_\alpha}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\ &\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).\end{aligned} \tag{4.27}$$

On the other hand, we also see that:

$$\begin{aligned}&LC^{No\Phi,div,I_1,\gamma}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] =\\ &\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).\end{aligned} \tag{4.28}$$

Lastly, to describe the fourth sublinear combination, we introduce some notation: For each $k, 2 \le k \le |I_1|$ we define $\hat{C}^{l,i_1\dots i_a}_g$ to stand for the sublinear combination which arises from $C^{l,i_1\dots i_a}_g$ by performing a cyclic permutation of the indices ${}_{i_k}, {}_{i_1}, {}_l$ (${}_{i_k}$ is picked out arbitrarily among ${}_{i_2},\dots,{}_{i_{|I_1|}}$) in the crucial factor $S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ijkl}$. We then conclude:

$$\begin{aligned}&LC^{No\Phi,div,I_1,\delta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] =\\ &-(|I_1|-1)Xdiv_{i_2}\dots Xdiv_{i_\alpha}\hat{C}^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\\ &+\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).\end{aligned} \tag{4.29}$$

Now, just by the first and second Bianchi identity we observe that:

$$
\begin{aligned}
&-(|I_1|-1)Xdiv_{i_2}\dots Xdiv_{i_\alpha}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\\
&-(|I_1|-1)Xdiv_{i_2}\dots Xdiv_{i_\alpha}\hat{C}^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\\
&=\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).
\end{aligned}
\tag{4.30}
$$

Next, we seek to analyze the sublinear combination:

$$
Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)].
$$

We have by definition that this sublinear combination can only arise by applying the last term in (2.2) to a pair of indices in the selected factor. We now break it into five sublinear combinations: We define the first sublinear combination to be the one that arises when we apply the fourth summand in (2.2) to a pair of indices $({}_{i_1},{}_b)$, where ${}_b$ is an original non-free index in the selected factor in $C^{l,i_1\dots i_a}_g$. We define the second sublinear combination to be one that arises by applying the last term in (2.2) to two free indices $({}_{i_1},{}_{i_k})$, $2\le k\le|I_1|$ in the selected factor. We define the third to be the one that arises by applying the last term in (2.2) to a pair $({}_{i_k},{}_b)$ where $k\ge 2$ and the index ${}_b$ is an original non-free index in the crucial factor. The fourth sublinear combination arises when we apply the last term in (2.2) to a pair $({}_{i_k},{}_{i_l})$ of free indices in the crucial factor, $2\le k,l\le|I_1|$. Lastly, the fifth sublinear combination is the one that arises by applying the last term in (2.2) to a pair of non-free original indices in $C^{l,i_1\dots i_a}_g$. We denote these sublinear combinations by $LC^{No\Phi,targ,B,\alpha}_{\phi_{u+1}}[\dots]$, $LC^{No\Phi,targ,B,\beta}_{\phi_{u+1}}[\dots]$, $LC^{No\Phi,targ,B,\gamma}_{\phi_{u+1}}[\dots]$, $LC^{No\Phi,targ,B,\delta}_{\phi_{u+1}}[\dots]$, $LC^{No\Phi,targ,B,\varepsilon}_{\phi_{u+1}}[\dots]$. It follows that:

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B,\alpha}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&-\frac{\nu\cdot\epsilon}{\nu+1}Xdiv_{i_2}\dots Xdiv_{i_\alpha}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\
&\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_a}C^{h,i_1\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1},
\end{aligned}
\tag{4.31}
$$

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B,\beta}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&-\frac{\nu\cdot(|I_1|-1)}{\nu+1}Xdiv_{i_2}\dots Xdiv_{i_\alpha}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\
&\sum_{t\in T}a_tXdiv_{i_1}\dots Xdiv_{i_\mu}C^{h,i_1\dots i_{a-1},i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),
\end{aligned}
\tag{4.32}
$$

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B,\gamma}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&-\frac{\nu\cdot(|I_1|-1)}{\nu+1}Xdiv_{i_2}\dots Xdiv_{i_\alpha}\hat{C}^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\
&\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_a}C^{h,i_1\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),
\end{aligned}
\tag{4.33}
$$

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B,\delta}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),
\end{aligned}
\tag{4.34}
$$

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B,\epsilon}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_a}C^{h,i_1\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).
\end{aligned}
\tag{4.35}
$$

Adding all the above we derive the first claim of our Lemma, (where the selected factor has a special free index $_{i_1} = {}_k$).

The second claim of our Lemma (where there is no special index in the crucial factor) follows more easily. We now have that $_{i_1}$ is not a special free index, so we will now consider all the sublinear combinations above where $_{i_1}$ is not mentioned, and also whenever we mentioned above one of the free indices $_{i_2},\dots,{}_{i_{|I_1|}}$ we will now read "one of the free indices $_{i_1},\dots,{}_{i_{|I_1|}}$" (since the index $_{i_1}$ is not special now). We then find that all the relevant equations will hold, with the exception of (4.29), (4.33), which now become:

$$
\begin{aligned}
&LC^{No\Phi,div,I_1,\delta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),
\end{aligned}
\tag{4.36}
$$

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B,\gamma}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_a}C^{h,i_1\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),
\end{aligned}
\tag{4.37}
$$

by application of the first and second Bianchi identity. This concludes the proof of our Lemma. □

Now, we study the sublinear combination (4.23) in the setting of Lemma 1.2. We recall the discussion from the introduction in [6] on the *crucial factor*.[79] We recall that the crucial factor is defined *in terms of the u-simple character* $\vec{\kappa}_{simp}$ by examining the tensor fields $C^{l,i_1\dots i_\mu}_g$, $l\in L^z, z\in Z'_{Max}$ (for a precise definition see the discussion above the statement of Lemma 1.1). Once it has been defined, we may speak of the crucial factor(s) for *any* tensor field with the u-simple character $\vec{\kappa}_{simp}$ (in fact, even for any complete contractions with a weak character $Weak(\vec{\kappa}_{simp})$). In each tensor field $C^{l,i_1\dots i_a}, l\in L$, we will denote by $\{T_1,\dots,T_M\}$ the set of crucial factors.

We will now separately consider the sublinear combinations in

$$\begin{aligned}&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\&LC^{No\Phi,div,I_1}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\end{aligned} \tag{4.38}$$

that have a factor $\nabla\phi_{u+1}$ contracting against $T_1,\dots,T_M$. We use the symbols $LC^{No\Phi,targ,B,T_i}$ and $LC^{No\Phi,div,I_1,T_i}$ to illustrate that we are considering those sublinear combinations. Again, we will denote by

$$\sum_{t\in T}a_tC^{t,i_1\dots i_{\mu-1},i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}$$

a generic linear combination of *acceptable* $(\mu-1)$-tensor fields which have a simple character $\vec{\kappa}^+_{simp}$ but are doubly subsequent to each $\vec{L^z}, z\in Z'_{Max}$.

Definition 4.6 *Here, ϵ^l_i will stand for the number of derivative indices in the crucial factor $T_i=\nabla^{(m)}R_{ijkl}$ that are not free and are not contracting against a factor $\nabla\phi_h$. γ^l_i will stand for the number of indices in the other factors in $C^{l,i_1\dots i_a}_g$ that are not contracting against a factor $\nabla\phi_h$.*

Note: By abuse of notation, we will be writing ϵ_i,γ_i instead of ϵ^l_i,γ^l_i from now on). We claim:

Lemma 4.2 *If $C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ has one internal free index (say i_1) in the crucial factor $T_i=\nabla^{(m)}R_{i_1jkl}$, then:*

[79] Recall that in this setting the "crucial" and "selected" factors coincide.

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B,T_i}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&LC^{No\Phi,div,I_1,T_i}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&-(\gamma_i+\epsilon_i)Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\
&\sum_{t\in T}a_tXdiv_{i_1}\dots Xdiv_{i_{a-1}}C^{t,i_1\dots i_{a-1},i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_\mu}C^{h,i_1\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).
\end{aligned}
\tag{4.39}
$$

Moreover, in the case of Lemma 1.2 and if $C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ *has two internal free indices (say* $_{i_1}$ *and* $_{i_2}$*) in the crucial factor* $T_i=\nabla^{(m)}R_{i_1ji_2l}$*, then:*

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B,T_i}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&LC^{No\Phi,div,I_1,T_i}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&-(\gamma_i+\epsilon_i)Xdiv_{i_2}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\\
&-(\gamma_i+\epsilon_i)Xdiv_{i_1}Xdiv_{i_3}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_2}\phi_{u+1}+\\
&\sum_{t\in T}a_tXdiv_{i_1}\dots Xdiv_{i_{a-1}}C^{t,i_1\dots i_{a-1},i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_\mu}C^{h,i_1\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).
\end{aligned}
\tag{4.40}
$$

Finally, if the crucial factor T_i *has no internal free indices then:*

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B,T_i}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&LC^{No\Phi,div,I_1,T_i}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_\mu}C^{h,i_1\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).
\end{aligned}
\tag{4.41}
$$

Proof of Lemma 4.2:

The proof of this claim is similar to the previous one. We start with the case where the crucial factor T_i has one internal free index.

We again denote by ${}_{i_1}$ the one internal free index in the crucial factor T_i and we denote by ${}_{i_2}, \dots, {}_{i_{|I_1|}}$ the other free indices. We divide the sublinear combination $LC^{No\Phi,div,I_1}_{\phi_{u+1}}[\dots]$ into further sublinear combinations (indexed by ${}_{\alpha}, \dots, {}_{\epsilon}$) as in the previous case. We calculate:

$$LC^{No\Phi,div,I_1,\alpha}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] = \\ -\gamma C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1} + \sum_{q\in Q} a_q C^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}); \tag{4.42}$$

(we have used the first Bianchi identity here).

The second sublinear combination is a little more complicated.

$$LC^{No\Phi,div,I_1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] = \\ -(|I_1|-1)Xdiv_{i_2}\dots Xdiv_{i_\alpha}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+ \\ \sum_{q\in Q} a_q C^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}). \tag{4.43}$$

On the other hand, we also see that:

$$LC^{No\Phi,div,I_1,\gamma}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] = \\ \sum_{q\in Q} a_q C^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}). \tag{4.44}$$

Lastly, to describe the fourth sublinear combination, we introduce some notation: For each $k, 2\le k\le |I_1|$ we define $\hat{C}^{l,i_1\dots i_a}_g$ to stand for the sublinear combination which arises from $C^{l,i_1\dots i_a}_g$ by performing a cyclic permutation of the indices ${}_{i_k}, {}_{i_1}, {}_j$ in the crucial factor $\nabla^{(m)}R_{ijkl}$. We then have that:

$$LC^{No\Phi,div,I_1,\delta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] = \\ -(|I_1|-1)Xdiv_{i_2}\dots Xdiv_{i_\alpha}\hat{C}^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+ \\ \sum_{q\in Q} a_q C^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}); \tag{4.45}$$

(we have used the second Bianchi).

Now, just by the first and second Bianchi identity we derive that:

$$
\begin{aligned}
&-(|I_1|-1)Xdiv_{i_2}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\\
&-(|I_1|-1)Xdiv_{i_2}\dots Xdiv_{i_a}\hat{C}^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}=\\
&\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).
\end{aligned}
\tag{4.46}
$$

Now, we study the second sublinear combination:

$$
Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B,T_i}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)].
$$

By definition and by the transformation law (2.2) that this sublinear combination can only arise by applying the last term in (2.2) to a pair of indices in the crucial factor T_i. We now break it into five sublinear combinations: We define the first sublinear combination to be the one that arises when we apply the fourth summand in (2.2) to a pair of indices $({}_{i_1},{}_b)$, where ${}_b$ is an original non-free index in the crucial factor in $C^{l,i_1\dots i_a}_g$. We define the second sublinear combination to be the one that arises by applying the last term in (2.2) to two free indices $({}_{i_1},{}_{i_k})$, $2\le k\le|I_1|$ in the crucial factor T_i. We define the third to be the one that arises by applying the last term in (2.2) to a pair $({}_{i_k},{}_b)$ where $k\ge 2$ and the index ${}_b$ is an original non-free index in the crucial factor. The fourth is when we apply the last term in (2.2) to a pair $({}_{i_k},{}_{i_l})$ of free indices in the crucial factor, $2\le k,l\le|I_1|$. Lastly, the fifth sublinear combination is the one that arises by applying the last term in (2.2) to a pair of non-free original indices in $C^{l,i_1\dots i_a}_g$. We denote these sublinear combinations by $LC^{No\Phi,targ,B,T_i,\alpha}_{\phi_{u+1}}[\dots]$, $LC^{No\Phi,targ,B,T_i,\beta}_{\phi_{u+1}}[\dots]$, $LC^{No\Phi,targ,B,T_i,\gamma}_{\phi_{u+1}}[\dots]$, $LC^{No\Phi,targ,B,T_i,\delta}_{\phi_{u+1}}[\dots]$, $LC^{No\Phi,targ,B,T_i,\varepsilon}_{\phi_{u+1}}[\dots]$. It follows that:

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B,T_i,\alpha}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&-\epsilon_iXdiv_{i_2}\dots Xdiv_{i_\alpha}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\
&\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_a}C^{h,i_1\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1},
\end{aligned}
\tag{4.47}
$$

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B,T_i,\beta}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&-(|I_1|-1)Xdiv_{i_2}\dots Xdiv_{i_\alpha}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\
&\sum_{t\in T}a_tXdiv_{i_1}\dots Xdiv_{i_\mu}C^{h,i_1\dots i_{a-1},i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),
\end{aligned}
\tag{4.48}
$$

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B,T_i,\gamma}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&-(|I_1|-1)Xdiv_{i_2}\dots Xdiv_{i_\alpha}\hat{C}^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\
&\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_a}C^{h,i_1\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),
\end{aligned}
\tag{4.49}
$$

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B,T_i,\delta}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),
\end{aligned}
\tag{4.50}
$$

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B,T_i,\varepsilon}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_a}C^{h,i_1\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).
\end{aligned}
\tag{4.51}
$$

Adding all the above we obtain our conclusion in the case where the crucial factor T_i has precisely one special free index.

We now consider the case where the are two special free indices in T_i. We have assumed that these two special free indices are ${}_{i_1},{}_{i_2}$ in the crucial factor $T_i=\nabla^{(m)}R_{i_1ji_2l}$. We will moreover slightly alter our notational conventions: Now, we will still speak of the non-special free indices in the crucial factor, but they will in fact be ${}_{i_3},\dots,{}_{i_{|I_1|}}$. Moreover, in the above sublinear combinations when we referred to the indices ${}_{i_1}$ or ∇^{i_1} we will now read "one of the indices ${}_{i_1},{}_{i_2}$ or $\nabla^{i_1},\nabla^{i_2}$". Lastly, we define $LC^{No\Phi,div,I_1,T_i,\varepsilon}$ to stand for the sublinear combination that arises by applying the transformation law (2.2) to the pair of divergence indices $\nabla^{i_1},\nabla^{i_2}$ when they have hit the same factor.

We now want to describe our first sublinear combination. To do so, we need just a little more notation. We denote by $\tilde{C}^{l,i_1\dots i_a}_g$ the tensor field that arises by switching the indices ${}_{i_1}={}_{i},{}_{l}$ and by $\tilde{C'}^{l,i_1\dots i_a}_g$ the tensor field that arises by switching the indices ${}_{i_2}={}_{k},{}_{j}$. We calculate:

$$
\begin{aligned}
&LC^{No\Phi,div,I_1,T_i,\alpha}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&-(\gamma-1)Xdiv_{i_2}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\\
&-(\gamma-1)Xdiv_{i_1}Xdiv_{i_3}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_2}\phi_{u+1}\\
&-Xdiv_{i_2}\dots Xdiv_{i_a}\tilde{C}^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\\
&-Xdiv_{i_1}Xdiv_{i_3}\dots Xdiv_{i_a}\tilde{C'}^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_2}\phi_{u+1}+\\
&\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).
\end{aligned}
\tag{4.52}
$$

Here we note that if $a=\mu$ then the $(\mu-1)$-tensor fields $\tilde{C}^{l,i_1\dots i_a}_g$ and $\tilde{C'}^{l,i_1\dots i_a}_g$ will be *doubly subsequent* to the maximal refined double characters $\vec{L'^z}, z\in Z'_{Max}$.

The second sublinear combination is a little more complicated:

$$
\begin{aligned}
&LC^{No\Phi,div,I_1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&-(|I_1|-2)Xdiv_{i_2}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\\
&-(|I_1|-2)Xdiv_{i_1}Xdiv_{i_3}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_2}\phi_{u+1}\\
&+\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).
\end{aligned}
\tag{4.53}
$$

On the other hand, we also see that:

$$
\begin{aligned}
&LC^{No\Phi,div,I_1,T_i,\gamma}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).
\end{aligned}
\tag{4.54}
$$

To describe the fourth sublinear combination, we introduce some notation: For each $k, 2\le k\le |I_1|$ we define $\hat{C}^{l,i_1\dots i_a}_g$ to stand for the sublinear combination which arises from $C^{l,i_1\dots i_a}_g$ by performing a cyclic permutation of the indices ${}_{i_k},{}_{i_1},{}_j$ in the crucial factor $\nabla^{(m)}_{r_1\dots r_m}R_{ijkl}$. We also define $\hat{C'}^{l,i_1\dots i_a}_g$ to stand for the sublinear combination which arises from $C^{l,i_1\dots i_a}_g$ by performing a cyclic permutation of the indices ${}_{i_k},{}_{i_2},{}_l$ in the crucial factor $\nabla^{(m)}_{r_1\dots r_m}R_{ijkl}$. We then derive:

$$
\begin{aligned}
&LC^{No\Phi,div,I_1,T_i,\delta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&-(|I_1|-2)Xdiv_{i_2}\dots Xdiv_{i_\alpha}\hat{C}^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\\
&-(|I_1|-2)Xdiv_{i_1}Xdiv_{i_3}\dots Xdiv_{i_\alpha}\hat{C'}^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_2}\phi_{u+1}\\
&+\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_a}C^{h,i_1\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}\\
&+\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).
\end{aligned}
\tag{4.55}
$$

Lastly, in this case we compute:

$$
\begin{aligned}
&LC^{No\Phi,div,I_1,T_i,\varepsilon}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&-Xdiv_{i_2}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}-\\
&-Xdiv_{i_1}Xdiv_{i_3}\dots Xdiv_{i_\alpha}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_2}\phi_{u+1}.
\end{aligned}
\tag{4.56}
$$

Now, it is quite straightforward in this case to understand the sublinear combinations in

$$Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B,T_i}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)].$$

We calculate:

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B,T_i,\alpha}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&-\epsilon_iXdiv_{i_2}\dots Xdiv_{i_\alpha}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\
&-\epsilon_iXdiv_{i_1}Xdiv_{i_3}\dots Xdiv_{i_\alpha}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_2}\phi_{u+1}+\\
&\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_a}C^{h,i_1\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1},
\end{aligned}
\tag{4.57}
$$

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B,T_i,\beta}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&-(|I_1|-2)Xdiv_{i_2}\dots Xdiv_{i_\alpha}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\\
&-(|I_1|-2)Xdiv_{i_1}Xdiv_{i_3}\dots Xdiv_{i_\alpha}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_2}\phi_{u+1}+\\
&\sum_{t\in T}a_tXdiv_{i_1}\dots Xdiv_{i_\mu}C^{h,i_1\dots i_{a-1},i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),
\end{aligned}
\tag{4.58}
$$

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B,T_i,\gamma}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&-(|I_1|-2)Xdiv_{i_2}\dots Xdiv_{i_\alpha}\hat{C}^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\\
&-(|I_1|-2)Xdiv_{i_1}Xdiv_{i_3}\dots Xdiv_{i_\alpha}\hat{C'}^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_2}\phi_{u+1}+\\
&\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_a}C^{h,i_1\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),
\end{aligned}
\tag{4.59}
$$

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B,T_i,\delta}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),
\end{aligned}
\tag{4.60}
$$

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B,\epsilon}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_a}C^{h,i_1\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).
\end{aligned}
\tag{4.61}
$$

We thus derive the claim of our Lemma in the case where the crucial factor has two internal free indices, by adding all the above equations.

The last case, where the crucial factor has no internal free indices follows more easily. We now have that ${}_{i_1}$ is not a special free index, so we will now consider all the sublinear combinations above where ${}_{i_1}$ is not mentioned, an also whenever we mentioned above one of the free indices ${}_{i_2},\dots,{}_{i_{|I_1|}}$ we will now read "one of the free indices ${}_{i_1},\dots,{}_{i_{|I_1|}}$" (since the index ${}_{i_1}$ is not special now). We then have that all the relevant equations will hold, with the exception of (4.29), (4.33), which now become:

$$
\begin{aligned}
&LC^{No\Phi,div,I_1,\delta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),
\end{aligned}
\tag{4.62}
$$

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B,\gamma}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_a}C^{h,i_1\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),
\end{aligned}
\tag{4.63}
$$

by application of the first or second Bianchi identity. □

We are now in a position to plug in all the equations from this section into (6.2) and derive Lemma 1.1:

Proof of Lemma 1.1: We use the equations (6.2), (4.11), (4.19) and plug in all the other equations from this section, and also use (3.48). We write out our conclusion concisely:

$$\begin{aligned}
&-\sum_{z\in Z'_{Max}}\sum_{l\in L^z} a_l(\gamma+\epsilon-1-2(s-1)-X)Xdiv_{i_2}\dots Xdiv_{i_a}\\
&C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\
&\sum_{t\in T} a_t Xdiv_{i_2}\dots Xdiv_{i_\mu}C^{t,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\
&\sum_{h\in H} a_h Xdiv_{i_1}\dots Xdiv_{i_\mu}C^{h,i_1\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{q\in Q} a_q C^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})=0,
\end{aligned}\tag{4.64}$$

modulo complete contractions of length $\geq \sigma+u+2$. Here

$$\sum_{t\in T} a_t Xdiv_{i_2}\dots Xdiv_{i_\mu}C^{t,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}$$

stands for a generic linear combination of $(\mu-1)$-tensor fields with a $(u+1)$-simple character $\vec{\kappa}^+_{simp}$ but that are doubly subsequent to each $\vec{L^z}'$, $z\in Z'_{Max}$.

Moreover, by the definition of weight we derive the following elementary identity:

$$\begin{aligned}
&(\text{Total number of indices in each complete contraction in}\\
&Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u))\\
&=n+2s=\gamma+\epsilon+2|\Phi|+2|I_1|+|I_2|+1.
\end{aligned}\tag{4.65}$$

Thich shows us that the quantity $(\gamma+\epsilon-1-2(s-1)-X)$ is *fixed* for each $l\in L^z, z\in Z'_{Max}$; (i.e. the same in all the terms in the first line in (4.64)).

A counting argument shows that $(\gamma+\epsilon-1-2(s-1)-X)=0$ if and only if $\sigma_1=0$ (i.e. there are no factors $\nabla^{(m)}R_{ijkl}$ in $\vec{\kappa}_{simp}$), and the tensor fields $C^{l,i_1\dots i_\mu}_g$, $l\in L^z$ have exactly one "exceptionnal index"; we define an index $_b$ in $C^{l,i_1\dots i_g}_g$ (in the form (1.5) to be exceptional if it belongs to a factor $S*\nabla^{(\nu)}R_{ijkl}$ or $\nabla^{(B)}\Omega_h$, is non-special, and moreover: if it belongs to the crucial factor $S_*\nabla^{(\nu)}R_{ijkl}$ then it must be non-free and not contracting against a factor $\nabla\phi_h$; if it belongs to a non-crucial factor then it must not contracting against a factor $\nabla\phi_h$. Notice that by weight considerations, if one of the tensor fields

$C_g^{l,i_1\dots i_\mu}$, $l\in L^z, z\in Z'_{Max}$ has exactly one exceptional index, then all of them do.

We will check that Lemma 1.1 indeed holds in this very special case (we will call it the *unfortunate case*) in a Mini-Appendix at the end of this paper. In all the remaining cases, we derive Lemma 1.1 by just dividing (4.64) by $(\gamma+\epsilon-1-2(s-1)-X)$.

Proof of Lemma 1.2: We again use equations (6.1) and (4.11) and replace according to all the equations of this subsection, also using (3.29), (3.30). Then, we first consider the case where the maximal refined double characters $\vec{L^z}, z\in Z'_{Max}$ have 2 free indices ${}_i={}_{i_1}, {}_k={}_{i_2}$ in the crucial factor(s), and we denote by M the number of crucial factors and by ${}_{i_1}, {}_{i_2}, {}_{i_3}, {}_{i_4},\dots, {}_{i_{2M-1}}, {}_{i_{2M}}$ the special free indices that belong to those factors (${}_{i_1}, {}_{i_2}$ are the indices ${}_i, {}_k$ in the first crucial factor etc.) we deduce:

$$\begin{aligned}
&-\sum_{z\in Z'_{Max}}\sum_{l\in L^z}\sum_{i_h\in I_{l,*}} a_l(\gamma_i+\epsilon_i-1-2(s-1)-X)Xdiv_{i_1}\dots X\hat{div}_{i_h}\dots Xdiv_{i_\mu}\\
&\tilde{C}_g^{l,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\\
&+\sum_{t\in T} a_t Xdiv_{i_2}\dots Xdiv_{i_\mu} C_g^{t,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\
&\sum_{h\in H} a_h Xdiv_{i_1}\dots Xdiv_{i_\mu} C_g^{h,i_1\dots i_\mu,i_*}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{q\in Q} a_q C_g^q(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})=0,
\end{aligned}\tag{4.66}$$

modulo complete contractions of length $\geq\sigma+u+2$. Here again

$$\sum_{t\in T} a_t C_g^{t,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}$$

is a generic linear combination of $(\mu+1)$-tensor fields with $(u+1)$-simple character $\vec{\kappa}^+_{simp}$ and a refined double character that is subsequent to each $\vec{L^z}'$, $z\in Z'_{Max}$. Moreover, the same elementary observation as above continues to hold, hence we deduce that $(\gamma_i+\epsilon_i-2(s-1)-X)$ is independent of $l\in L^z$ (and of the choice of crucial factor), and also that it is non-zero (by a counting argument again–we use the fact that $\gamma\geq 2$). Thus we derive Lemma 1.2 by dividing by this constant.

Finally, consider the case where the maximal refined double characters $\vec{L^z}, z\in Z'_{Max}$ have one special free index in the crucial factor(s). Recall that $I_{l,*}$ stands for the index set of the special free indices that belong to the (one of the) crucial factor(s). We deduce:

$$
\begin{aligned}
&-\sum_{z\in Z'_{Max}}\sum_{l\in L^z}\sum_{i_h\in I_{l,*}} a_l(\gamma+\epsilon-2(s-1)-X)Xdiv_{i_1}\dots X\hat{div}_{i_h}\dots Xdiv_{i_\mu}\\
&\tilde{C}^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\
&\sum_{t\in T} a_t Xdiv_{i_2}\dots Xdiv_{i_a} C^{t,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\
&\sum_{h\in H} a_h Xdiv_{i_1}\dots Xdiv_{i_\mu} C^{h,i_1\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{q\in Q} a_q C^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})+\sum_{z\in Z} a_z C^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})=0,
\end{aligned}
\tag{4.67}
$$

modulo complete contractions of length $\geq \sigma+u+2$. Moreover, the same elementary observation as above continue to hold, hence we deduce that $(\gamma_i+\epsilon_i+-2(s-1)-X)$ is independent of l and also that it is non-zero (since in this case $\gamma\geq 3$). This shows Lemma 1.2 in this case also. $\Box$

4.2 Mini-Appendix: Proof of Lemma 1.1 in the unfortunate case.

In order to show Lemma 1.1 in this setting, we recall that here $\sigma_1=0$ and $X=0$ (hence all factors $\nabla^{(B)}\Omega_x$ must be contracting against some factor $\nabla\phi'_h$). We also observe that by weight considerations all tensor fields in (1.6) *other than* the ones indexed in L^z, for a given $z\in Z'_{Max}$ can have at most $M-1$ free indices belonging to the crucial factor $S_*\nabla^{(\nu)}R_{ijkl}\nabla^i\tilde{\phi}_1$.

We make a further observation: If for some $C^{l,i_1\dots i_\mu}_g$ the exceptionnal index belongs to a factor $\nabla^{(2)}\Omega_h$ (with exactly two derivatives), or to a simple factor S_*R_{ijkl}, then all tensor fields $C^{l,i_1\dots i_\mu}_g$ must have that property; this follows by weight considerations. We call this subcase A. The other case we call subcase B.

Let us prove Lemma 1.1 in subcase B, which is the hardest: We observe that by explicitly constructing divergences of $(\mu+1)$-tensor fields (indexed in H below), as allowed in Lemma 1.1, we can write:

$$
\begin{aligned}
&\sum_{l\in L^z} a_l C^{l,i_1\dots i_\mu}_g = Xdiv_{i_{\mu+1}}\sum_{h\in H} a_h C^{h,i_1\dots i_{\mu+1}}_g+\\
&\sum_{l\in\tilde{L}^z} a_l C^{l,i_1\dots i_\mu}_g+\sum_{j\in J} a_j C^{j,i_1\dots i_\mu}_g.
\end{aligned}
\tag{4.68}
$$

Here the terms indexed in J are simply subsequent to $\vec{\kappa}_{simp}$. The terms indexed in $\tilde{L}^z$ have all the features of the terms in L^z in the LHS (in particular the have the same, maximal, refined double character), and in addition:

1. The exceptional index does not belong to the crucial factor.

2. If the index ${}_l$ in the crucial factor contracts against a special index in a factor $T = S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ijkl'}$, then it contracts against the index ${}_k$; moreover, if the exceptional index belongs to the factor T (say it is the index ${}_{r_\nu}$, then the indices ${}_l, {}_{r_\nu}$ are symmetrized over.

3. If the index ^{l} against which the index ${}_l$ in the crucial factor contracts is a non-special index (and hence we may assume wlog that it is a derivative index–denote it by ∇^l) then it *does not* belong to some specified factor T' in $\vec{\kappa}_{simp}$.

In view of the above, we may assume that all the tensor fields indexed in L^z in (1.6) have the features described above. Let us then break the index set L^z into two subsets: We say that $l \in L^z_1$ if and only if the index ${}_l$ in the crucial factor contracts against a special index. We set $L^z_2 = L^z \setminus L^z_1$.

We will then prove:

$$\sum_{l\in L^z_1} a_l C^{l,(i_1\dots i_\mu)}_g = 0, \tag{4.69}$$

$$\sum_{l\in L^z_2} a_l C^{l,(i_1\dots i_\mu)}_g = 0. \tag{4.70}$$

Clearly, if we can proe the above then Lemma 1.1 will follow in subcase B of the unfortunate case.

Proof of (4.69): For future reference, let us denote by $L^z_{1,b} \subset L^z_1$ the index set of tensor fields for which the index ${}_l$ in the crucial factor contracts against the factor $S_*\nabla^{(\nu')}R_{i'j'k'l'}\nabla^{i'}\tilde{\phi}_b$.

The main tool we will use in this proof will be used for the other claims in this subsection. We consider $Image^1_Y[L_g] = 0$ (the first conformal variation of (1.6), and we pick out the sublinear combination $Image^{1,*}_Y[L_g]$ of terms with length $\sigma + u$, with the crucial factor $S_*\nabla^{(\nu)}R_{ijkl}$ being replaced by a factor $\nabla^{(\nu+2)}Y$, and the factor $\nabla\phi_1$ contracting against some other factor. We derive that:

$$Image^{1,*}_Y[L_g] = 0,$$

modulo complete contractions of length $\sigma+u+1$. We further break up $Image^{1,*}_Y[L_g]$, into sublinear combinations $Image^{1,*,b}_Y[L_g]$, where a complete contraction belongs to $Image^{1,*,b}_Y[L_g]$ if and only if $\nabla\phi_i$ contracts against the factor $S_*\nabla^{(\nu')}R_{ijkl}\nabla^i\tilde{\phi}_b$. Clearly, we also have:

$$Image^{1,*}_Y[L_g] = 0,$$

modulo complete contractions of length $\sigma + u + 1$. Now, in the above equation we formally replace the two factors $\nabla_a\phi_1, \nabla_c\phi_b$ by a factor g_{ac}. This

clearly produces a new true equation. We then act on the new true equation by $Ricto\Omega_{p+1}$,[80] thus deriving a new true equation:

$$Z_g(\Omega_1,\dots,\Omega_p,\Omega_{p+1},\phi_1,\dots,\phi_u)=0, \tag{4.71}$$

which holds modulo terms of length $\sigma+u+1$.

This equation can in fact be re-expressed in a more useful form: Let us denote by $L_*\subset L_\mu\bigcup L_{>\mu}$ the index set of terms for which one special index (say the index $_l$ in the crucial factor contracts against a special index (say $_{l'}$) in the factor $S_*\nabla^{(\nu')}R_{i'j'k'l'}$.[81] We denote by $\overline{C}^{l,i_1\dots i_a}_g(\dots,Y,\Omega_{p+1})$ the tensor field that arises from $C^{l,i_1\dots i_\mu}$ by replacing the expression $S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ijkl}\otimes S_*\nabla^{(\nu')}_{t_1\dots t_{\nu'}}R_{i'j'k'}{}^l\otimes\nabla^i\tilde{\phi}_1\otimes\nabla^{i'}\tilde{\phi}_b$ by $\nabla^{(\nu+2)}_{r_1\dots r_\nu jk}Y\otimes\nabla^{(\nu'+2)}_{t_1\dots t_{\nu'}j'k'}\Omega_{p+1}$. Denote by $Cut[\vec{\kappa}_{simp}]$ the simple character of the resulting tensor field. Then (4.71) can be re-expressed as:

$$\sum_{l\in L_*}a_lXdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_\mu}_g(\dots,Y,\Omega_{p+1})+\sum_{j\in J}a_jC^j=0. \tag{4.72}$$

modulo complete contractions of length $\sigma+u+1$. The terms indexed in J are simply subsequent to $Cut[\vec{\kappa}_{simp}]$. We now apply the Eraser to all factors $\nabla\phi_h$ that contract against the factor $\nabla^{(B)}Y$; by abuse of notation, we still denote the resulting tensor fields, complete contractions etc by $C^{l,i_1\dots i_\mu}_g, C^j_g$.

Now, we apply the inverse integration by part to the above equation,[82] deriving an integral equation:

$$\int_{M^n}\sum_{l\in L_*}a_l\hat{C}^l_g(\dots,Y,\Omega_{p+1})+\sum_{j\in J}a_jC^jdV_g=0. \tag{4.73}$$

Here the complete contractions $\hat{C}^l_g(\dots,Y,\Omega_{p+1})$ arise from the tensor fields $C^{l,i_1\dots i_\mu}_g(\dots,Y,\Omega_{p+1})$ by making the free indices into internal contractions.[83] We then consider the silly divergence formula for the above, which arises by integrating by parts wrt. $\nabla^{(B)}Y$; denote the resulting equation by

$$silly_Y[\sum_{l\in L_*}a_l\hat{C}^l_g(\dots,Y,\Omega_{p+1})+\sum_{j\in J}a_jC^j].$$

We consider the sublinear combination $silly_{Y,*}[\dots]$ which consists of terms with length $\sigma+u$, and $\mu-M$ internal contractions and all factors $\nabla\phi_h$ differentiated only once. Clearly, $silly_{Y,*}[\dots]=0$. This equation can be re-expressed in the form:

[80] See the Appendix in [3].

[81] Notice that $L^z\subset L_*$.

[82] This has been defined at in section 3 of [7].

[83] For future reference, let us write $\sum_{l\in L^z}a_l\hat{C}^l_g=\Delta^MY\sum_{l\in L^z}a_lC'^l_g$.

$$\sum_{l\in L^z_{1,b}} a_l Spread^M_{\nabla,\nabla}[C'^l_g] + \sum_{j\in J} a_j C^j_g = 0.$$

Here $Spread_{\nabla,\nabla}$ stands for an operation that hits a pair of different factors (not in the form $\nabla\phi$) by derivatives ∇^a, ∇_a that contract against each other and then adding over the resulting terms. From the above, we derive that:

$$\sum_{l\in L^z_{1,b}} a_l C'^l_g + \sum_{j\in J} a_j C^j_g = 0.$$

Now, we formally replace all $\mu - M$ internal contractions in the above by factors $\nabla\upsilon$ (this gives rise to a new true equation), and then formally replacing the expression $Y \otimes S_*\nabla^{(\nu')}_{t_1\dots t_{\nu'}j'k'}\Omega_{p+1}$ by $S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ij(free)l} \otimes \nabla^i\tilde{\phi}_1 \otimes S_*\nabla^{(\nu')}_{t_1\dots t_{\nu'}}R_{i'j'k'}{}^l\nabla^{r_1}\phi_{x_1}\dots\nabla^{x_z}\phi_z$, where the indices ${}_{r_1},\dots,{}_{r_\nu},{}_j$ that are not contracting against a factor $\nabla\phi_x$ are free.[84] The resulting true equation is our claim of Lemma 1.1 for the index set $L^z_{1,b}$. Thus we have derived (4.69).

The proof of (4.70) follows by an adaptation of the argument above. We consider the equation $Image^1_Y[L_g] = 0$ and pick out the sublinear combination $Image^{1,*}_Y[L_g]$ where $\nabla\phi_1$ contracts against the factor T'. Clearly $Image^{1,*}_Y[L_g] = 0$. (Among the tensor fields indexed in L^z, such terms can only arise from the terms in $L_g = 0$ where the $Xdiv_k$ for the index ${}_k$ in the crucial factor is forced to hit the factor T'; then the factor $S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ijkl}\nabla^i\tilde{\phi}_1$ must be replaced by $\nabla^{(\nu+2)}_{r_1\dots r_\nu jl}Y\nabla_k\phi_1$). Now, from $Image^{1,*}_Y[L_g] = 0$, we again repeat the argument with the inverse integration by parts and the silly divergence formula, picking out in that equation the sublinear combination with $\mu - M$ internal contractions and all functions ϕ_h differentiated only once, with the factor $\nabla\phi_1$ contracting against the factor T' and with M particular contractions between the factor T' and another factor T''. We derive that this sublinear combination, say $silly_+[\dots]$ must vanish separately; $silly_+[\dots] = 0$. It is also in one-to-one correspondence with the sublinear combination $\sum_{l\in L^z_2} a_l C^{l,(i_1\dots i_\mu)}_g$, by the same reasoning as above. We can then reproduce the formal operations as above, and derive the claim of Lemma 1.1 for the index set L^z_2.

Proof of Lemma 1.1 in subcase A of the unfortunate case: We again consider the equation $Image^1_Y[L_g] = 0$ and pick out the sublinear combination $Image^{1,*}_Y[L_g] = 0$ where $\nabla\phi_1$ contracts against the factor T'. This sublinear combination must vanish separately, thus we derive a new true equation, which we again denote by $Image^{1,*}_Y[L_g] = 0$. Now, we again apply the inverse integration by parts (replacing the $Xdiv$'s by internal contractions), deriving an integral equation. From this new integral equation, we derive a silly divergence formula, by integrating by parts with respect to the factor $\nabla^{(B)}Y$. We

[84] We add factors $\nabla\phi_x$ and free indices to the first factor according to the form of the crucial factor in the tensor fields $C^{l,i_1\dots i_\mu}_g$, $l \in L^z$ in (1.6).

pick out the sublinear combination of terms with length $\sigma + u$, $\mu - M$ internal contractions, all functions ϕ_h differentiated once, the factor $\nabla\phi_1$ contracting against T', where T' contains no exceptional indices, and where there are M exceptional indices on some factor T'', contracting against M exceptional indices in some other factor $T'' \neq T'$. Again, this sublinear combination vanishes separately, and is in one-to-one correspondence with the sublinear combination $\sum_{l\in L^z} a_l C_g^{l,(i_1\dots i_\mu)}$, by the same reasoning as above. We can then reproduce the formal operations as above, and derive the claim of Lemma 1.1 for the index set L_2^z. □

4.3 Mini-Appendix: A postponed claim.

We directly derive Proposition 1.1 when the tensor fields of maximal refined double character in (1.6) are in the special forms described at the end of the introduction.

Proof of the postponed claim: Observe that by weight considerations, all tensor fields in (1.6) will have rank μ. Also, none will have special free inices in a factor S_*R_{ijkl}.

For each $l \in L_\mu$ we denote by $C_g^{l,i_1|A}$ the vector field that arises by replacing the factor $S_*R_{ijkl}\nabla^i\tilde{\phi}_1$ by $\nabla^{(2)}_{jk}Y\nabla_l\phi_1$; we also denote by $C_g^{l,i_1|A}$ the vector field that arises by replacing the factor $S_*R_{ijkl}\nabla^i\tilde{\phi}_1$ by $-\nabla^{(2)}_{jl}Y\nabla_k\phi_1$. Denote the $(u-1)$-simple character of the resulting tensor fields by $\vec{\kappa}'_{simp}$. We then derive an equation:

$$\sum_{l\in L_\mu} a_l Xdiv_{i_1}\dots Xdiv_{i_\mu}[C_g^{l,i_1\dots i_\mu|A} + C_g^{l,i_1\dots i_\mu|B}] = 0.$$

Now, we apply Lemma 4.10 in [6] to the above[85] and derive:

$$\sum_{l\in L_\mu} a_l[C_g^{l,i_1\dots i_\mu|A} + C_g^{l,i_1\dots i_\mu|B}]\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon = 0.$$

Finally, formally replacing the expression $\nabla^{(2)}_{(ab)}Y\nabla_c\phi_1$ by $S_*R_{i(ab)c}\nabla^1\tilde{\phi}_1$ we derive our claim. □

5 Part B: A proof of Lemas 3.3, 3,4, 3.5 in [6]

The rest of this paper is devoted to proving Lemma 1.3 and also Lemmas 3.3, 3.4 in [6]. In this proof we will derive the strongest consequence of the local equation (1.6), which is the common assumption of all three Lemmas above. This consequence, a *new, less complicated* local equation is called the "grand conclusion". In case A of Lemma 1.6, the "grand conclusion" coincides with the claim of Lemma 1.3.

[85] There are no special free indices in factors $S_*\nabla^{(\nu)}R_{ijkl}$ here, hence no danger of "forbidden cases".

In order to derive Lemma 1.3 in case B, in section 9 we will apply the grand conclusion (and certain other equations derived below) in less straightforward ways. In the next section we provide more technical details regarding the derivation of the "grand conclusion".

5.1 A general discussion regarding the derivation of the "grand conclusion":

The derivation of the "grand conclusion" can be divided into two parts: In the first part, we repeat the analysis performed in [8]: We consider the first conformal variation $Image^1_{\phi_{u+1}}[L_g]=0$ of (1.6).[86] Since the terms in the LHS of the equation $L_g=0$ can be written out as general complete contractions of the form (1.8), the first conformal variation can be calculated by virtue of the transformation laws of the curvature tensor and the Levi-Civita connection under conformal changes of the metric, 2.1), (2.2).

Now as part A, in $Image^1_{\phi_{u+1}}[L_g]=0$, we pick out the sublinear combination $Image^{1,+}_{\phi_{u+1}}[L_g]$ of terms which either *are* or *can give rise to*[87] terms of the type that appear in the claim of Lemma 1.3. As we have shown in [8], this sublinear combination will vanish separately, modulo junk terms which we may disregard:

$$Image^{1,+}_{\phi_{u+1}}[L_g]+\sum_{z\in Z}a_zC^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})=0. \tag{5.1}$$

In section 6 we repeat the analysis of (5.1) that we performed in [8]. (This analysis involves much calculation, but also the appropriate application of the inductive assumption of Proposition 1.1). The resulting equation, however, completely fails to give us the desired conclusion of Lemma 1.3.[88]

In section 7, we seek to analyze a *second* equation which arises in the conformal variation $Image^1_{\phi_{u+1}}[L_g]=0$ of (1.6), and which we had previously discarded: We pick out the sublinear combination $Image^{1,\beta}_{\phi_{u+1}}[L_g]$ which consists of terms with an *internal contraction* (see subsection 7.1 below for more details). Again, this sublinear combination vanishes separately, modulo junk terms which we may disregard:

$$Image^{1,\beta}_{\phi_{u+1}}[L_g]+\sum_{z\in Z}a_zC^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})=0. \tag{5.2}$$

In principle, one might think that this new equation should be of no interest, since the claim of Lemma 1.3 involves terms with no internal contractions. However, we are able to suitably analyze (5.2), to derive a new equation

[86] Recall that is general, the first conformal variation of a true equation $L_g(\psi_1,\dots,\psi_t)=0$ that depends on an auxilliary Riemannian metric g is defined through the formula: $Image^1_\phi[L_g(\psi_1,\dots,\psi_t)]:=\frac{d}{dt}|_{t=0}L_{e^{2t\phi}}(\psi_1,\dots,\psi_t)$; of course $Image^1_\phi[L_g(\psi_1,\dots,\psi_t)]=0$.

[87] After application of the curvature identity.

[88] The defficiencies of the analysis of equation (5.1) are explained at the end of subsection 6.2.

$Im^{1,\beta}_{\phi_{u+1}}[L_g]=0$ below (see 7.61)), which, combined with the analysis of (5.1) yields the "grand conclusion", as follows:

We define a simple formal operation $Soph\{\dots\}$ which acts on the terms in $Im^{1,\beta}_{\phi_{u+1}}[L_g]=0$ to produce terms of the type that appear in the claim of Lemma 1.3. We then *add* the resulting equation to our analysis of (5.1) and observe many miraculous cancellations of "bad terms". The resulting local equations are collectively called the "grand conclusion"; they are the equations (8.3), (8.4), (8.5) below.

Recall Language Conventions: Firstly, we recall that our hypothesis is equation (1.6). We recall that in that equation all the tensor fields have a given simple character $\vec{\kappa}_{simp}$ and are acceptable and all the complete contractions C^j_g have a weak character $Weak(\vec{\kappa}_{simp})$ (they are not assumed to be acceptable).

We recall that a free index that is of the form ${}_i, {}_j, {}_k, {}_l$ in some factor $\nabla^{(m)}R_{ijkl}$ or ${}_k, {}_l$ in some factor $S_*\nabla^{(\nu)}R_{ijkl}$ is called special.

Next, we recall the discussion regarding the "selected factor". Let us firstly re-explain how the various *factors* in the various complete contractions and tensor fields in (1.6) can be distinguished: We recall that for all the tensor fields and complete contractions appearing in (1.6) and for each $\nabla\phi_f, 1\le f\le u$, there will be a unique factor $\nabla^{(m)}R_{ijkl}$, $S_*\nabla^{(\nu)}R_{ijkl}$ or $\nabla^{(B)}\Omega_h$ against which $\nabla\phi_f$ is contracting. Therefore, for each $\nabla\phi_f, 1\le f\le u$ we may unambiguously speak of *the* factor against which $\nabla\phi_f$ is contracting for each of the tensor fields and contractions in (1.6). Furthermore, for each $h, 1\le h,\le p$ we may also unambiguously speak of the factor $\nabla^{(B)}\Omega_h$ in each tensor field and contraction in (1.6).

On the other hand, we may have factors $\nabla^{(m)}R_{ijkl}$ in the terms in (1.6) that are not contracting against any factors $\nabla\phi_h$. We notice that there is the same number of such factors in each of the tensor fields and the complete contractions in (1.6) (since they all have the same weak character $Weak(\vec{\kappa}_{simp})$). We sometimes refer to such factors as "generic factors of the form $\nabla^{(m)}R_{ijkl}$".

We recall the two cases that we have distinguished in the setting of Lemma 1.3: Recall that we have denoted by M the number of free indices in the critical factor for the tensor fields $C^{l,i_1\dots i_\mu}_g$, $l\in\bigcup_{z\in Z'_{Max}}L^z$ (i.e. the tensor fields with the maximal refined double character $\vec{L}^z$, for a given $z\in Z'_{Max}$). We also denote by $M'(=\alpha)$ the number of free indices in the second critical factor for the tensor fields $C^{l,i_1\dots i_\mu}_g, l\in\bigcup_{z\in Z'_{Max}}L^z$. We recall that case A is when $M'\ge 2$, and case B is when $M'\le 1$.

The "special subcase" of case B: We introduce a "special subcase" of case B, in which case the derivation of "grand conclusion" will be somewhat different. We say that (1.6) falls under the "special subcase" when the tensor fields of maximal refined double character in (1.6) are in the form:

$$\begin{aligned}&contr(\nabla_{(free)}R_{\sharp\sharp\sharp\sharp}\otimes R_{\sharp\sharp\sharp\sharp}\otimes\dots\otimes R_{\sharp\sharp\sharp\sharp}\otimes\\&S_*R_{ix\sharp\sharp}\otimes\dots\otimes S_*R_{ix\sharp\sharp}\otimes\nabla^{(2)}_{y\sharp}\Omega_1\otimes\dots\otimes\nabla^{(2)}_{y\sharp}\Omega_p\otimes\nabla\phi_1\otimes\dots\otimes\nabla\phi_u).\end{aligned}\tag{5.3}$$

(In the above, each index $_\sharp$ must contract against another index in the form $_\sharp, _x, _y$; the indices $_x$ are either contracting against another index $_\sharp, _x, _y$ or are free, and the indices $_y$ are either contracting against indices $_\sharp, _x, _y$ or are free or contract against a factor $\nabla\phi_h$). In some instances, our argument will be modified to treat those "special subcases".

Now, we will be deriving three equations below, (8.3), (8.4), (8.5), which will be collectively called the "grand conclusion". We next explicitly spell out the hypotheses under which the grand conclusion will be derived in the list below: The main assumption will be equation (1.6), and we also have the extra assumptions stated in Lemma 1.3; we re-iterate that list of assumptions below.

1. In both cases A and B, no tensor field of rank μ can have an internal free index in any factor $\nabla^{(m)}R_{ijkl}$, nor a free index of the form $_k, _l$ in any factor $S_*\nabla^{(\nu)}R_{ijkl}$ (this is the main assumption of Lemma 1.3).
2. In both cases A and B, there are no μ-tensor fields in (1.6) with a free index in the form $_j$ in some factor S_*R_{ijkl}, with no derivatives. THis is a re-statement of the assumption $L^+_\mu = \emptyset$.
3. In both cases A and B, no $(\mu+1)$-tensor field in (1.6) contains a factor S_*R_{ijkl} with two internal free indices (this is a re-statement of the assumption $L''_+ = \emptyset$).

Now, in order to describe the terms that appear in the RHSs of our equations below, we recal some notational conventions:

We introduce the notion of a "contributor", adapted to this setting:

Definition 5.1 *In the setting of Lemma 1.3, $\sum_{h\in H} a_h C^{h,i_1\dots i_a}_g$ will stand for a generic linear combination of a-tensor fields ($a \geq \mu$) with a u-simple character $\vec{\kappa}_{simp}$, a weak $(u+1)$-character $Weak(\vec{\kappa}^+_{simp})$[89] and the following additionnal features: Either the tensor fields above are acceptable, or they are unacceptable with one unacceptable factor $\nabla\Omega_x$, which either contracts against $\nabla\phi_{u+1}$ or does not contract against any $\nabla\phi_h$. Furthermore, if a tensor field $C^{h,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}$ has $a=\mu$ and one unacceptable factor $\nabla\Omega_x$ which does not contract against $\nabla\phi_{u+1}$ then it must have a $(u+1)$-simple character $\vec{\kappa}^+_{simp}$,[90] and moreover $\nabla\phi_{u+1}$ must be contracting against a derivative index, and moreover if it is contracting against a factor $\nabla^{(B)}\Omega_x$ then $B\geq 3$. Moreover, if $a=\mu$ and the factor $\nabla\phi_{u+1}$ is contracting against $\nabla\Omega_h$ then we require that at least one of the μ free indices should be non-special, and there should be a removable index in the tensor field $C^{i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}$. Finally, if a tensor field*

[89] $\vec{\kappa}^+_{simp}$ is some chosen $(u+1)$-simple character where $\nabla\phi_{u+1}$ is *not* contracting against a special index.

[90] In other words $\nabla\phi_{u+1}$ is not contracting against a special index.

$C_g^{i_1\dots i_a,i_*}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}$ *has* $a=\mu$ *but the factor* $\nabla\phi_{u+1}$ *is contracting against a special index then all* μ *free indices must be non-special.*

We will be calling the tensor fields in those linear combinations "contributors".

We recall that $\sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$ stands for a generic linear combination of complete contractions of length $\sigma+u+1$ in the form (1.8) which are simple subsequent to the u-simple character $\vec{\kappa}_{simp}$.[91]

Let us introduce one new piece of notation that will be useful further down. We will define a generic linear combination of $(\mu-1)$-tensor fields which appears in the grand conclusion only in case B, in the special subcase.

Definition 5.2 *We denote by*

$$\sum_{b\in B'} a_b C_g^{b,i_1\dots i_{\mu-1}i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_\mu}\phi_{u+1}$$

a generic linear combination of acceptable $(\mu-1)$*-tensor fields with length* $\sigma+u+1$*, with a u-simple character* $\vec{\kappa}_{simp}$*, and with the factor* $\nabla\phi_{u+1}$ *contracting against a non-special index in the selected factor and moreover each of the* $(\mu-1)$ *free indices belong to a different factor.*

Moreover, in the special subcase when in addition $\mu=1$*, we additionaly require that the index* ${}_{i_\mu}$ *should be a derivative index, and moreover if it belongs to a factor* $\nabla^{(B)}\Omega_h$ *then* $B\geq 3$*. Furthermore, we require that if we change the selected factor from* T_a *to* T_b*, then* $\sum_{b\in B'} a_b C_g^{b,i_1}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ *changes by erasing the index* ∇_{i_1} *from* T_a *and adding a derivative* ∇_{i_1} *onto* T_b*.*

6 The first half of the "grand conclusion".

As explained, our first step in deriving the "grand conclusion" is to repeat the analysis of the equation (5.1) from [8]. We refer the reader to subsection 2.2 in [8] for the strict definition of the sublinear combination $Image^{1,+}_{\phi_{u+1}}[L_g]$. *We recall that the sublinear combination* $Image^{1,+}_{\phi_{u+1}}[L_g]$ *in* $Image^{1}_{\phi_{u+1}}[L_g]$ *is defined once we have picked a (set of) selected factor(s) in* $\vec{\kappa}_{simp}$. We also recall that the equation (5.1) has been proven to hold (modulo complete contractions of length $\geq\sigma+u+2$).[92]

We also recall (from the end of subsection 2.2 in [8]) the natural break-up of the sublinear combination $Image^{1,+}_{\phi_{u+1}}[L_g]$ into three "pieces" according to

[91] Recall that this means that one of the factors $\nabla\phi_h$ which are supposed to contract against an index ${}_i$ in a factor $S_*\nabla^{(\nu)}R_{ijkl}$ in the simple character $\vec{\kappa}_{simp}$ is now contracting against a derivative index in a factor $\nabla^{(m)}R_{ijkl}$.

[92] We recall that the "junk terms" $\sum_{z\in A} a_z C_g^z$ have length $\sigma+u+1$ and a factor $\nabla^{(A)}\phi_{u+1}$ with $A\geq 2$.

which *rule* of conformal variation a given term has arisen from:

$$Image^{1,+}_{\phi_{u+1}}[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] = CurvTrans[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] \\ + LC[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] + W[L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]. \tag{6.1}$$

(The sublinear combination $CurvTrans[L_g\dots]$ consists of terms with length $\sigma+u$ and a factor $\nabla^{(A)}\phi_{u+1}, A\ge 2$, while the sublinear combinations $LC[L_g], W[L_g]$ consist of terms with length $\sigma+u+1$ and a factor $\nabla\phi_{u+1}$). We then recall the study of the sublinear combination $CurvTrans[L_g\dots]$ that we performed in [8]:[93]

We recall that the sublinear combination $CurvTrans[L_g]$ will be *zero, by definition* if the selected factor is of the form $\nabla^{(A)}\Omega_h$. We also recall that if the selected factor is of the form $\nabla^{(m)}R_{ijkl}$ or $S_*\nabla^{(\nu)}R_{ijkl}$, then we have proven that the sublinear combination $CurvTrans[L_g]$ in (6.1) can be expressed as in equations (3.31) and (3.47) in [8], respectively;[94] *and moreover we proved in [8] that these equations also hold in the setting of Lemma 1.3.* We also recall that the terms inside parentheses in (3.31) and (3.47) are defined to be *zero* in the setting of Lemma 1.3. The result of this analysis in [8], as a new local equation, namely equation (4.1) in that paper, which we reproduce here:

$$CurvTrans^{study}[L_g] + LC[L_g] + W[L_g] = 0; \tag{6.2}$$

this holds modulo complete contractions of length $\ge \sigma+u+2$.

The purpose of the first half of this paper will be to study the sublinear combinations $LC_{\phi_{u+1}}[\dots], W_{\phi_{u+1}}[\dots]$ in equation (6.2).

Now, our aim in this section will be to study the sublinear combinations $LC[L_g]$ and $W[L_g]$ in the context of Lemma 1.3 from [8]. We recall the Lemma 2.2 in [8], which has also been proven in the setting of Lemma 1.3. In view of this, we only need to study the sublinear combinations $LC^{No\Phi}[L_g]$ and $W[L_g]$.

Our aim here is to recall our description of the term $CurvTrans^{study}[L_g]$ appearing in (6.2),[95] and then the understand the sublinear combinations $LC^{No\Phi}[L_g], W[L_g]$ in (6.1), in the setting of Lemma 1.3.

6.1 Proof of Lemma 1.3: A description of the sublinear combination $CurvTrans[L_g]$ in this setting.

In this subsection we just reproduce the equations (3.31) and (3.48) from part A, for the reader's convenience. These equations describe the sublinear combination $CurvTrans^{study}[L_g]$ appearing in (6.2). Recall that (3.31) and (3.47)

[93] As noted there, the analysis also applies in the setting of Lemma 1.3.

[94] We re-express the conclusions of equations (3.31) and (3.47) in new notation in the next subsection.

[95] We recall that this analysis was performed in [8].

in [8]) correspond to the cases where the selected factor(s) is (are) in the form $S_*\nabla^{(\nu)}R_{ijkl}$, $\nabla^{(m)}R_{ijkl}$. We recall that if the selected factor is in the form $\nabla^{(A)}\Omega_h$ *then by definition* $CurvTrans[L_g]=CurvTrans^{study}[L_g]=0$.

Notation: If the selected factor is of the form $S_*\nabla^{(\nu)}R_{ijkl}$, then for each $l\in L$ we denote by I^l_1 the index set of free indices that belong to the selected factor T_l, where $T_l = S_*\nabla^{(\nu_l)}_{r_1\dots r_\nu}R_{ijkl}$ (note there are ν_l derivatives on the selected factor). By our Lemma hypothesis (the assumption 1 in the list of the previous subsection), each of the free indices ${}_i\in I^l_1$ will be one of the indices ${}_{r_1},\dots,{}_{r_\nu},{}_j$ in the factor T_l and $\nu_l>0$. Then, equation (3.47) in [8] can be re-expressed as:

$$\begin{aligned}
&\sum_{l\in L}a_l CurvTrans[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&\sum_{j\in J}a_j CurvTrans[C^j_g(\Omega_1\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&\sum_{l\in L_\mu}a_l\frac{1}{\nu_l}\sum_{i_s\in I^l_1}Xdiv_{i_1}\dots\hat{Xdiv}_{i_s}\dots Xdiv_{i_\mu}C^{l,i_1\dots i_\mu}_g(\Omega_1\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_s}\phi_{u+1}\\
&+\sum_{h\in H}a_h Xdiv_{i_1}\dots Xdiv_{i_a}C^{h,i_1,\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{j\in J}a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})+\sum_{z\in Z}a_z C^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).
\end{aligned}\tag{6.3}$$

(In fact, in this case the tensor fields indexed in H are also acceptable–but we will not be using this fact).

On the other hand, when the selected factor is of the form $\nabla^{(m)}R_{ijkl}$, then (3.31) in [8] can be re-expressed in the form:

$$\begin{aligned}
&\sum_{l\in L}a_l CurvTrans[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&\sum_{j\in J}a_j CurvTrans[C^j_g(\Omega_1\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&\sum_{h\in H}a_h Xdiv_{i_1}\dots Xdiv_{i_a}C^{h,i_1,\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\
&\sum_{j\in J}a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})+\sum_{z\in Z}a_z C^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).
\end{aligned}\tag{6.4}$$

(In this case also the tensor fields indexed in H are acceptable, but this will not matter).

6.2 A study of the sublinear combinations $LC^{No\Phi}[L_g]$, $W[L_g]$ in the setting of Lemma 1.3.

As explained in the beginning of this section, our aim is to find analogues of the equations concerning the sublinear combinations[96] $LC^{No\Phi}[L_g]$, $W[L_g]$ in [8].

It is again immediate by the definitions that for each $j \in J$ in (1.6) we must have:

$$W[C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)] = \sum_{j\in J} a_j C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_{u+1}), \tag{6.5}$$

$$LC^{No\Phi}[C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)] = \sum_{j\in J} a_j C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_{u+1}); \tag{6.6}$$

(using generic notation on the right hand side).

So the challenge is again to understand the two sublinear combinations:

$$LC^{No\phi}[Xdiv_{i_1} \dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)],$$

$$W[Xdiv_{i_1} \dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)].$$

We again divide the two linear combinations above into the same linear combinations as in section 3 in part A. Thus, our aim here is to find analogues of the equations (4.12), (4.13), (**??**), (4.15), (4.16), (4.18), (4.20), (4.20) in that paper to find analogues of Lemmas 4.1 and 4.2 and to find an analogue of those Lemmas when the selected factor is of the form $\nabla^{(A)}\Omega_s$.

Important Point: In the cases where the selected factor is a curvature term (in the form $\nabla^{(m)}R_{ijkl}$ or $S_*\nabla^{(\nu)}R_{ijkl}$), the major difference with the setting of Lemmas 3.1 and 3.2 in [6] (which were proven in [8]) is the role of the contractions that belong to the linear combinations $\sum_{q\in Q} \dots$. Recall that in [6] $\sum_{q\in Q} \dots$ stood for a *generic* linear combination of complete contractions with length $\sigma + u + 1$ for which the factor $\nabla\phi_{u+1}$ was contracting against a *non-special*[97] index in the selected factor (which was in one of the forms $\nabla^{(m)}R_{ijkl}$, $S_*\nabla^{(\nu)}R_{ijkl}$).

In the settings of Lemmas 1.1, 1.2, such generic complete contractions were *simply subsequent* to the $(u+1)$-simple character $\vec{\kappa}^+_{simp}$ that we were interested in, and hence they could be *disregarded* (since they were *allowed* in the conclusion of our Lemma). In this setting of Lemma 1.3, the complete contractions indexed in $\sum_{q\in Q} \dots$ will have precisely a $(u+1)$-simple character $\vec{\kappa}^+_{simp}$ that we are interested in. Hence they *can not* be disregarded, as they were in part A of this paper.

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[96](which appear in (6.2))

[97]Recall that a special index is an index of the form ${}_{i}, {}_{j}, {}_{k}, {}_{l}$ in a factor $\nabla^{(m)}R_{ijkl}$, or an index of the form ${}_{k}, {}_{l}$ in a factor $S_*\nabla^{(\nu)}R_{ijkl}$.

In this sense, our aim for this subsection will be to obtain a precise understanding of the terms belonging to the sublinear combinations $\sum_{q\in Q}\dots$ in the right hand sides of equations (4.12), (4.13), (4.14), (4.15), (4.16), (4.18), (4.20), (4.22) in [8] and in the conclusions of Lemmas 4.1 and 4.2 there.

A study of the sublinear combination $W[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g]$ in this context: The analogues of equations (4.13), (4.14), (4.15), (4.16), (4.18), (4.20), (4.20) in the setting of Lemma 1.3.

For each $C^{l,i_1\dots i_a}_g$ we will denote by $\{T_1,\dots,T_{b_l}\}$ the set of selected factors. Also, for each selected factor T_i we will denote by $I^{T_i}_1$ the set of free indices that belong to T_i and by $I^{T_i}_2$ the set of free indices that *do not* belong to T_i. We recall that by our Lemma hypotheses, if a factor $\nabla^{(m)}R_{ijkl}$ in $C^{l,i_1\dots i_\mu}_g$, $l\in L_\mu$ has free indices then those free indices *must* be derivative indices. Also, if a factor $S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ijkl}$ has free indices, they must be of the form ${}_{r_1},\dots,{}_{r_\nu},{}_j$. We then consider any $C^{l,i_1\dots i_\mu}_g$, $l\in L_\mu$ with a free index in the selected factor T_i; we then calculate the analogue of (4.13), (4.14), (4.16):

$$\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_\mu}W^{targ,T_i}[C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&W^{targ,T_i,div}[Xdiv_{i_1}\dots Xdiv_{i_\mu}C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&2\sigma^*\sum_{i_h\in I^{T_i}_1}Xdiv_{i_1}\dots X\hat{d}iv_{i_h}\dots Xdiv_{i_\mu}C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_h}\phi_{u+1}\\
&+\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_\mu}C^{h,i_1\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}\\
&+\sum_{z\in Z}a_zC^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1});
\end{aligned}\tag{6.7}$$

here σ^* stands for $\sigma_1+\sigma_2$ if the crucial factor is of the form $\nabla^{(p)}\Omega_h$ and for $(\sigma_1+\sigma_2-1)$ if it is in any other form.

On the other hand, if $C^{l,i_1\dots i_a}_g$ has $a=\mu$ and does not have free indices in the selected factor T_i, or if $a>\mu$, we derive the analogue of (4.15):

$$\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}W^{targ,T_i}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&W^{targ,T_i,div}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_\mu}C^{h,i_1\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}\\
&+\sum_{z\in Z}a_zC^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).
\end{aligned}\tag{6.8}$$

The first hard case is to find the analogue of equation (4.20). Some notation will prove useful to do this:

Definition 6.1 *We define $(I_2^{T_i})^{2,dif}$ to stand for the set of pairs of indices $({}_{i_k}, {}_{i_l})$, where ${}_{i_k}, {}_{i_l} \in I_2^{T_i}$, for which ${}_{i_k}$ and ${}_{i_l}$ do not belong to the same factor. Then, for each pair $({}_{i_k}, {}_{i_l}) \in (I_2^{T_i})^{2,dif}$, we define*

$$[C_g^{l,i_1\dots i_a,i_*|T_i}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)g^{i_k i_l}]\nabla_{i_*}\phi_{u+1}$$

to stand for the $(a-2)$-tensor field that arises from $C_g^{l,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ by contracting the indices ${}_{i_k}, {}_{i_l}$ against each other and adding a derivative index ∇_{i_*} on the selected factor T_i, and then contracting it against a factor $\nabla_{i_*}\phi_{u+1}$ (and also if the selected factor is of the form $S_*\nabla^{(\nu)}R_{ijkl}$ performing an extra S_*-symmetrization).

It then follows that the sublinear combination $\sum_{q\in Q} a_q C_g^q(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$ in (4.20) is precisely of the form:

$$-\sum_{i=1}^{b_l}\sum_{(i_k,i_l)\in I_2^{T_i,2,def}} Xdiv_{i_1}\dots X\hat{d}iv_{i_k}\dots X\hat{d}iv_{i_l}\dots Xdiv_{i_a} [C_g^{l,i_1\dots i_a,i_*|T_i}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)g^{i_k i_l}\nabla_{i_*}\phi_{u+1}]. \tag{6.9}$$

Next, we seek to understand the term $\sum_{q\in Q} a_q C_g^q(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$ in $Xdiv_{i_1}\dots Xdiv_{i_a}\{LC_{\phi_{u+1}}^{No\Phi,targ,A}[C_g^{l,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}$, see (4.22). With our new notation, we easily observe that:

$$Xdiv_{i_1}\dots Xdiv_{i_a}\{LC_{\phi_{u+1}}^{No\Phi,targ,A}[C_g^{l,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\} = \sum_{h\in H} a_h Xdiv_{i_1}\dots Xdiv_{i_a} C_g^{h,i_1\dots i_a,i_*}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}. \tag{6.10}$$

A harder task is to understand the sublinear combination $\sum_{q\in Q} a_q C_g^q(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$ in (4.18) and (4.12).

Analogues of (4.18) and (4.12):

Analogue of (4.18): We recall that (by definition) $LC_{\phi_{u+1}}^{No\Phi,free}[C_g^{l,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ is a linear combination that consists of tensor fields in the form:

$$C_g^{t,i_1\dots\hat{i}_f\dots i_a}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_f}\phi_{u+1}, \tag{6.11}$$

where the index ${}_{i_f}$ (which is a free index) belongs to the factor $\nabla\phi_{u+1}$. We observe (by virtue of the transformation law (2.2)) that the tensor fields $C_g^{t,i_1\dots\hat{i}_f\dots i_a}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ will then be acceptable except possibly for one unacceptable factor $\nabla\Omega_h$. For each $l \in L$, we will write out that linear combination as:

$$\sum_{t\in T^l} a_t C^{t,i_1\dots\hat{i}_f\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{f_t}}\phi_{u+1}. \tag{6.12}$$

We then observe that (4.18) again holds, where the sublinear combination $\sum_{q\in Q} a_q C^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$ is in fact of the form:

$$\begin{aligned}&\sum_{t\in T^l} a_t Xdiv_{i_1}\dots X\hat{d}iv_{i_{f_t}}\dots Xdiv_{i_a}\{\nabla^{i_{f_t}}_{sel}[C^{t,i_1\dots\hat{i}_{f_t}\dots i_a}_g\\&(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{f_t}}\phi_{u+1}]\};\end{aligned} \tag{6.13}$$

here

$$\{\nabla^{i_{f_t}}_{sel}[C^{t,i_1\dots\hat{i}_{f_t}\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{f_t}}\phi_{u+1}]\}$$

stands for the $(a-1)$-tensor field that arises from $C^{t,i_1\dots\hat{i}_{f_t}\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{f_t}}\phi_{u+1}$ by hitting the selected factor (if it is unique) with a derivative index $\nabla^{i_{f_t}}$, or if there are multiple selected factors $T_i, i=1,\dots,b_l$ then:

$$\begin{aligned}&\{\nabla^{i_{f_t}}_{sel}[C^{t,i_1\dots\hat{i}_{f_t}\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{f_t}}\phi_{u+1}]\}=\\&\sum_{i=1}^{b_l}\{\nabla^{i_{f_t}}_{T_i}[C^{t,i_1\dots\hat{i}_{f_t}\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{f_t}}\phi_{u+1}]\}\end{aligned} \tag{6.14}$$

(where $\nabla^{i_{f_t}}_{T_i}$ means that $\nabla^{i_{f_t}}$ is forced to hit the factor T_i).

We observe, that if $l\in L\setminus L_\mu$ then the right hand side in (6.13) is a contributor.[98] If $l\in L_\mu$, we must understand it in more detail:

We recall that if an index ${}_{i_y}\in I^{T_i}_2$ belongs to a factor $\nabla^{(m)}R_{ijkl}$, then by the hypothesis of Lemma 1.3, ${}_{i_y}$ must be a derivative index. If it belongs to a factor $S_*\nabla^{(\nu)}R_{ijkl}$, ${}_{i_y}$ must be one of the indices ${}_{r_1},\dots,{}_{r_\nu},{}_j$. For each ${}_{i_y}\in I_2$ that belongs to a factor $\nabla^{(m)}R_{ijkl}$, we denote by $\sigma(i_y)$ the number of indices in that factor that are *not* contracting against a factor $\nabla\phi_w$, *minus one*. We also define $C^{l,i_1\dots\hat{i}_y\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ to stand for the vector field that arises from $C^{l,i_1\dots i_\mu}_g$ by formally erasing the derivative index ${}_{i_y}$.

For each ${}_{i_y}\in I^{T_i}_2$ that belongs to a factor $\nabla^{(A)}\Omega_h$ we denote by $\sigma(i_y)$ the number of indices in that factor that are *not* contracting against a factor $\nabla\phi_y$, *minus one*. We again define $C^{l,i_1\dots\hat{i}_y\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ to stand for the tensor field that arises from $C^{l,i_1\dots i_\mu}_g$ by erasing the derivative index ${}_{i_y}$.

For each ${}_{i_y}\in I^{T_i}_2$ that belongs to a factor $T=S_*\nabla^{(\nu)}R_{ijkl}$, we denote by $\sigma(i_y)$ the number $[(\epsilon_T-1)+\frac{2\nu}{\nu+1}]$ (recall that ϵ_T stands for the number of indices ${}_{r_1},\dots,{}_{r_\nu},{}_j$ in that factor that *are not* contracting against a factor $\nabla\phi_f$) notice that if $\nu=0$ this number is zero). Now, we also define

[98] See Definition 5.1

$C_g^{l,i_1\dots\hat{i}_y\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ as follows: If $\nu\ge 1$, we assume for convenience that ${}_{i_y}$ is a derivative index (wlog because of the S_*-symmetrization). We then define $C_g^{l,i_1\dots\hat{i}_y\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ to stand for the $(\mu-1)$-tensor field that arises from $C_g^{l,i_1\dots i_\mu}$ by replacing the factor $S_*\nabla^{(\nu)}_{i_y r_2\dots r_\nu}R_{ijkl}$ by $S_*\nabla^{(\nu-1)}_{r_2\dots r_\nu}R_{ijkl}$. If $\nu=0$, we define $C_g^{l,i_1\dots\hat{i}_y\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ to be zero.

Applying (2.2), it then follows that the analogue of (4.12) in this setting will be:

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,free}_{\phi_{u+1}}[C_g^{l,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&-\sum_{i=1}^{b_l}\sum_{i_y\in I_2^{T_i}}\sigma(i_y)Xdiv_{i_1}\dots X\hat{div}_{i_y}\dots Xdiv_{i_a}[\nabla^{i_y}_{T_i}C_g^{l,i_1\dots\hat{i}_y\dots i_a}\\
&(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_y}\phi_{u+1}]+\sum_{z\in Z}a_zC^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1});
\end{aligned}
\tag{6.15}
$$

(as noted above, if $a>\mu$ then the second line is a contributor).

Analogue of (4.12): In order to determine the sublinear combination $\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ in (4.12), we will start with the case $l\in L_\mu$. We define a number $\tau(i_y)$ for each ${}_{i_y}\in I_2^{T_i}$: Firstly, if ${}_{i_y}$ belongs to a factor $\nabla^{(A)}\Omega_h$, we define $\tau(i_y)=0$. If ${}_{i_y}$ belongs to a factor $\nabla^{(m)}R_{ijkl}$ (and by the hypothesis of Lemma 1.3 it must be a derivative index), we define $\tau(i_y)=2$. Finally, if ${}_{i_y}$ belongs to a factor $S_*\nabla^{(\nu)}R_{ijkl}$ (and by our Lemma hypothesis it must be one of the indices ${}_{r_1},\dots,{}_{r_\nu},{}_j$), then we define $\tau(i_y)=\frac{2\nu}{\nu+1}$. We then observe that if $l\in L_\mu$, the sublinear combination $\sum_{q\in Q}a_qC^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ in (4.12) will be of the form:

$$
\begin{aligned}
&\sum_{i=1}^{b_l}\sum_{i_y\in I_2}\tau(i_y)Xdiv_{i_1}\dots X\hat{div}_{i_y}\dots Xdiv_{i_\mu}[\nabla^{i_y}_{T_i}C_g^{l,i_1\dots\hat{i}_y\dots i_\mu}(\Omega_1,\dots,\Omega_p,\\
&\phi_1,\dots,\phi_u)\nabla_{i_y}\phi_{u+1}],
\end{aligned}
\tag{6.16}
$$

while if $l\in L\setminus L_\mu$, it will simply be of the form:

$$\sum_{h\in H}a_hC_g^{h,i_1\dots i_a,i_*}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}.$$

Analogue of Lemmas 4.1, 4.2 in [8]:

Finally, we want to find a version of Lemmas 4.1 and 4.2 in this context, and also to find a version of these two Lemmas in the case where the selected factor

is of the form $\nabla^{(A)}\Omega_h$. As above, this boils down to understanding the sublinear combination $\sum_{q\in Q} a_q C^q_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ that appear in the statements of those Lemmas. In order to understand this sublinear combination, several more pieces of notation are needed:

Definition 6.2 *Consider any tensor field $C^{l,i_1\dots i_a}_g$ in the form (1.5) with a simple character $\vec{\kappa}_{simp}$.*

Firstly, for each selected factor T_i and for each pair of free indices $({}_{i_k},{}_{i_l})$ with ${}_{i_k}\in I^{T_i}_1, {}_{i_l}\in I^{T_i}_2$ we denote by $\nabla^{i_*}_{T_i}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)g^{i_k i_l}]\nabla_{i_*}\phi_{u+1}$ the $(a-2)$-tensor field that formally arises from $C^{l,i_1\dots i_a}_g$ by contracting the index ${}_{i_k}$ against ${}_{i_l}$ and then adding a derivative index ∇_{i_*} on the selected factor T_i and contracting it against a factor $\nabla_{i_*}\phi_{u+1}$.

Furthermore, we denote by $F_1,\dots,F_{\sigma-1}$ the set of real factors other than T_i,[99] and other than any $\nabla\phi_w$. If $|I^{T_i}_1|\ge 2$, we define $I^{T_i,2,*}_1$ to stand for the set of pairs $({}_{i_k},{}_{i_l})$ for which ${}_{i_k},{}_{i_l}\in I^{T_i}_1$, ${}_{i_k}\neq{}_{i_l}$ and at least one of the two indices (say ${}_{i_k}$ with no loss of generality) is a derivative index. If $|I^{T_i}_1|\le 1$ we define $I^{T_i,2,*}_1=\emptyset$.

For each $({}_{i_k},{}_{i_l})\in I^{T_i,2,*}_1$ and each $S, 1\le S\le\sigma-1$, we then additionally define:

$$\tilde{C}^{l,i_1\dots,\hat{i}_k\dots,i_a,i_z|S}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_l}\phi_{u+1} \tag{6.17}$$

to stand for the $(a-1)$-tensor field that arises from $C^{l,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ by first erasing the free index ${}_{i_k}$ (which is a derivative index), then contracting ${}_{i_l}$ against a factor $\nabla\phi_{u+1}$, and finally hitting the S^{th} real factor by a derivative free index ∇_{i_z}.

Now, to understand the sublinear combinations:

$$\begin{aligned}&Xdiv_{i_1}\dots Xdiv_{i_\mu}LC^{No\Phi,targ,T_i,B}_{\phi_{u+1}}[C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\&LC^{No\Phi,div,T_i,I_1}_{\phi_{u+1}}Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\end{aligned} \tag{6.18}$$

in this context, we distinguish three subcases: Either the selected factor(s) is (are) of the form $\nabla^{(m)}R_{ijkl}$, or of the form $S_*\nabla^{(\nu)}R_{ijkl}$, or of the from $\nabla^{(A)}\Omega_h$. We start with the first subcase, where the selected factor(s) is (are) of the form $\nabla^{(m)}R_{ijkl}$.

One more piece of notation:

Definition 6.3 *For each selected factor T_i we denote by $m^\sharp_i$ the number of derivative indices in the selected factor $T_i=\nabla^{(m)}R_{ijkl}$ that are not contracting against a factor $\nabla\phi_f$ and are not free. Also, recall (from part A) that γ_i stands for the number of indices in $C^{l,i_1\dots i_\mu}_g$ that do not belong to the selected factor T_i and are not contracting against factors $\nabla\phi_h$.*

[99] Recall that the "real factors" are in one of the forms $\nabla^{(m)}R_{ijkl}, S_*\nabla^{(\nu)}R_{ijkl}, \nabla^{(B)}\Omega_x$.

We then compute that for each $l \in L_\mu$, where the selected factor T_i is of the form $T_i = \nabla^{(m_i)} R_{ijkl}$:

$$Xdiv_{i_1} \dots Xdiv_{i_\mu} LC^{No\Phi,targ,T_i,B}_{\phi_{u+1}}[C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+$$
$$LC^{No\Phi,div,T_i,I_1}_{\phi_{u+1}} Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] = -\sum_{i_h\in I^{T_i}_1}$$
$$[2(m^\sharp_i+2)+\gamma_i] Xdiv_{i_1}\dots X\hat{div}_{i_h}\dots Xdiv_{i_\mu} C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_h}\phi_{u+1}-$$
$$3\sum_{(i_k,i_l)\in I^{T_i,2,*}_1} Xdiv_{i_1}\dots X\hat{div}_{i_l}\dots Xdiv_{i_\mu} C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_l}\phi_{u+1}$$
$$+\sum_{(i_k,i_l)\in I^{T_i,2,*}_1}\sum_{S=1}^{\sigma-1} Xdiv_{i_1}\dots X\hat{div}_{i_k}\dots X\hat{div}_{i_l}\dots Xdiv_{i_\mu} Xdiv_{i_z}$$
$$\tilde{C}^{l,i_1\dots,\hat{i}_k\dots,i_\mu,i_z|S}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_l}\phi_{u+1}$$
$$-\sum_{i_k\in I^{T_i}_1, i_l\in I^{T_i}_2} Xdiv_{i_1}\dots X\hat{div}_{i_k}\dots X\hat{div}_{i_l}\dots Xdiv_{i_\mu}$$
$$\nabla^{i_*}_{sel}[C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u) g^{i_k i_l}]\nabla_{i_*}\phi_{u+1}$$
$$+\sum_{h\in H} a_h Xdiv_{i_1}\dots Xdiv_{i_a} C^{h,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}. \tag{6.19}$$

Explanation of the calculations that bring out (6.19): The expression multiplied by $-2(m^\sharp_i+2)$ arises in two ways: Firstly, by applying the third summand in (2.2) to pairs $(\nabla_{i_k}, {}_b)$ where ${}_b$ is an original non-free index in $C^{l,i_1\dots,i_a}_g$ that is contracting against the crucial factor, and also (when ${}_b = {}_i, {}_j, {}_k, {}_l$) applying the second Bianchi identity twice. The second way is by applying the last summand in (2.2) to two indices $({}_{i_k}, {}_b)$ where both ${}_{i_k}, {}_b$ belong to the selected factor and then "completing the divergence" that we have created. With this observation, the rest of the sublinear combinations on the right hand side can be checked by book-keeping.

By the same analysis, if $l \in L\setminus L_\mu$, we calculate:

$$\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a} LC^{No\Phi,targ,T_i,B}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&LC^{No\Phi,div,I_1}_{\phi_{u+1}} Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] =\\
&\sum_{h\in H} a_h C^{h,i_1\dots i_{a-1},i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}-\\
&\sum_{i_k\in I^{T_i}_1, i_l\in I^{T_i}_2} Xdiv_{i_1}\dots X\hat{div}_{i_k}\dots X\hat{div}_{i_l}\dots Xdiv_{i_a}\\
&\nabla^{i_*}_{T_i}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u) g^{i_k i_l}]\nabla_{i_*}\phi_{u+1}.
\end{aligned} \tag{6.20}$$

The analogues of these two equations in the case where the selected factor is of the form $S_*\nabla^{(\nu)}R_{ijkl}$ or $\nabla^{(A)}\Omega_h$ are straightforward (recall that in this setting there is a unique selected factor so we will write I_1, I_2 rather than $I_1^{T_i}, I_2^{T_i}$). We only have to recall the trivial formula:

$$\nabla^{(m)}_{r_1\dots r_m}R_{ir_{m+1}kl} = S_*\nabla^{(m)}_{r_1\dots r_m}R_{ir_{m+1}kl} + \sum \nabla^{(m)}_{ir_1\dots r_{m-1}}R_{r_{m-1}r_mkl} + \sum Q(R), \tag{6.21}$$

where $\sum Q(R)$ stands for a generic linear combination of quadratic expressions in curvature, while $\sum \nabla^{(m)}_{ir_1\dots r_{m-1}}R_{r_mr_{m+1}kl}$ stands for a generic linear combination of tensors $\nabla^{(m)}R_{abcd}$ where the index $_i$ is a derivative index.

Analogue of Lemmas 4.1, 4.2 when the selected factor is of the form $S_*\nabla^{(\nu)}R_{ijkl}$ or $\nabla^{(A)}\Omega_h$: We first consider the case where the selected factor is of the form $S_*\nabla^{(\nu)}R_{ijkl}$ (in this case it will be unique) and we use (6.21). For the selected factor $T = \nabla^{(\nu)}_{r_1\dots r_\nu}R_{ijkl}$ we will denote by $\nu^\sharp$ the number of indices $_{r_1}, \dots, _{r_\nu}, _j$ that are not free and not contracting against a factor $\nabla\phi_h$.[100] Then, for each $l \in L_\mu$, we calculate:

$$\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B}_{\phi_{u+1}}[C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&LC^{No\Phi,div,I_1}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_\mu}C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] =\\
&-\sum_{i_h\in I_1}[\gamma+(\nu^\sharp+1)+(\nu^\sharp+\frac{\nu}{\nu+1})]Xdiv_{i_1}\dots X\hat{div}_{i_h}\dots Xdiv_{i_\mu}\\
&C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_h}\phi_{u+1}]\\
&-3\sum_{(i_k,i_l)\in I_1^{2,*}} Xdiv_{i_1}\dots X\hat{div}_{i_l}\dots Xdiv_{i_\mu}C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_l}\phi_{u+1}\\
&+\sum_{(i_k,i_l)\in I_1^{2,*}}\sum_{S=1}^{\sigma-1}Xdiv_{i_1}\dots X\hat{div}_{i_k}\dots X\hat{div}_{i_l}\dots Xdiv_{i_\mu}Xdiv_{i_z}\\
&\tilde{C}^{l,i_1\dots,\hat{i}_k\dots,i_\mu,i_z|S}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\\
&\nabla_{i_l}\phi_{u+1}-\sum_{i_k\in I_1,i_l\in I_2}Xdiv_{i_1}\dots X\hat{div}_{i_k}\dots X\hat{div}_{i_l}\dots Xdiv_{i_\mu}\\
&\nabla^{i_*}_{sel}[C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)g^{i_ki_l}]\nabla_{i_*}\phi_{u+1}\\
&+\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_a}C^{h,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1},
\end{aligned} \tag{6.22}$$

while for each $l \in L\setminus L_\mu$, we again have (6.20). A small extra explanation for this case: The sublinear combination multiplied by $-(\nu^\sharp+1+\frac{\nu}{\nu+1})$ arises by

[100] We should write $\nu_l, \nu_l^\sharp$ to stress that these numbers depend on the tensor field $C^{l,i_1\dots i_\mu}_g$, $l \in L_\mu$. However, for simplicity of notation, we will not do so.

virtue of applying the last summand in (2.2) to a pair of indices in the selected factor, one of which is free and one of which is not. We see that this formula can only be applied to indices $({}_{i_k},{}_b)$ in the selected factor if at least one of these indices is a derivative index.

Finally, in the case where the selected factor is of the form $\nabla^{(A)}\Omega_h$, we denote by $A^\sharp$ the number of indices in $\nabla^{(A)}\Omega_h$ that are not free and not contracting against a factor $\nabla\phi_f$. We then derive that for each $l\in L$, $(a\ge\mu)$:

$$\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}LC^{No\Phi,targ,B}_{\phi_{u+1}}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&LC^{No\Phi,div,I_1}_{\phi_{u+1}}Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]=\\
&-\sum_{i_h\in I_1}(\gamma+2A^\sharp)Xdiv_{i_1}\dots X\hat{d}iv_{i_h}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots\\
&,\phi_u)\nabla_{i_h}\phi_{u+1}\\
&-3\sum_{(i_k,i_l)\in I_1^{2,*}}Xdiv_{i_1}\dots X\hat{d}iv_{i_l}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots\\
&,\phi_u)\nabla_{i_k}\phi_{u+1}\\
&+\sum_{(i_k,i_l)\in I_1^{2,*}}\sum_{S=1}^{\sigma-1}Xdiv_{i_1}\dots X\hat{d}iv_{i_k}\dots X\hat{d}iv_{i_l}\dots Xdiv_{i_\mu}Xdiv_{i_z}\\
&\tilde{C}^{l,i_1\dots,\hat{i}_k\dots,i_\mu,i_z|S}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_l}\phi_{u+1}\\
&-\sum_{i_k\in I_1,i_l\in I_2}Xdiv_{i_1}\dots X\hat{d}iv_{i_k}\dots X\hat{d}iv_{i_l}\dots Xdiv_{i_a}\\
&\nabla^{i_*}_{crit}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)g^{i_ki_l}]\nabla_{i_*}\phi_{u+1}\\
&+\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_a}C^{h,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}.
\end{aligned}\tag{6.23}$$

Finally, for each $l\in L\setminus L_\mu$, we again have (6.20).

Thus, having computed the sublinear combinations $LC^{No\Phi}[L_g], W[L_g]$ in the setting of Lemma 1.3, we will now substitute them into (6.2).

We first consider the case where the selected factor is of the form $S_*\nabla^{(\nu)}R_{ijkl}$. (Recall that in this setting there is only one selected factor for each $C^{l,i_1\dots i_\mu}_g$, $l\in L^z$, $z\in Z'_{Max}$). We then derive a local equation:

$$\sum_{l\in L_\mu} a_l\{-\sum_{i_h\in I_1}(\gamma+(2\nu^\sharp+1)+\frac{\nu}{\nu+1}-2(\sigma_1+\sigma_2-1)-X)$$
$$Xdiv_{i_1}\dots X\hat{div}_{i_h}\dots Xdiv_{i_\mu}C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_h}\phi_{u+1}$$
$$-3\sum_{(i_k,i_l)\in I_1^{2,*}} Xdiv_{i_1}\dots X\hat{div}_{i_k}\dots Xdiv_{i_\mu}C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_k}\phi_{u+1}$$
$$-\frac{|I_1|}{\nu+1}Xdiv_{i_2}\dots Xdiv_{i_\mu}C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+$$
$$\sum_{i_y\in I_2}(-\sigma(i_y)+\tau(i_y))Xdiv_{i_1}\dots X\hat{div}_{i_y}\dots Xdiv_{i_\mu}$$
$$[\nabla^{i_y}_{sel}C^{l,i_1\dots\hat{i}_y\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_y}\phi_{u+1}]+$$
$$\sum_{(i_k,i_l)\in I_1^{2,*}}\sum_{S=1}^{\sigma-1}Xdiv_{i_1}\dots X\hat{div}_{i_k}\dots X\hat{div}_{i_l}\dots Xdiv_{i_\mu}Xdiv_{i_z}$$
$$\tilde{C}^{l,i_1\dots,\hat{i}_k\dots,i_\mu,i_z|S}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_l}\phi_{u+1}\}$$
$$-\sum_{l\in L}a_l\sum_{(i_k,i_l)\in I_2^{2,def}}Xdiv_{i_1}\dots X\hat{div}_{i_k}\dots X\hat{div}_{i_l}\dots Xdiv_{i_a}$$
$$[C^{l,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)g^{i_ki_l}]\nabla_{i_*}\phi_{u+1}$$
$$-\sum_{l\in L}a_l\sum_{i_k\in I_1,i_l\in I_2}Xdiv_{i_1}\dots X\hat{div}_{i_k}\dots X\hat{div}_{i_l}\dots Xdiv_{i_a}$$
$$\nabla^{i_*}_{sel}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)g^{i_ki_l}]\nabla_{i_*}\phi_{u+1}+$$
$$\sum_{j\in J}a_jC^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})+$$
$$\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_a}C^{h,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}=0, \tag{6.24}$$

modulo complete contractions of length $\geq\sigma+u+2$.

In the case where the selected factor(s) is (are) of the form $\nabla^{(m)}R_{ijkl}$, we derive:

$$\sum_{l\in L_\mu} a_l\{-\sum_{i=1}^{b_l}\sum_{i_h\in I_1^{T_i}}(\gamma_i+2(m_i^\sharp+2)-2(\sigma_1+\sigma_2-1)-X)$$
$$Xdiv_{i_1}\dots X\hat{d}iv_{i_h}\dots Xdiv_{i_\mu}C_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_h}\phi_{u+1}$$
$$-3\sum_{(i_k,i_l)\in I_1^{T_i,2,*}} Xdiv_{i_1}\dots X\hat{d}iv_{i_l}\dots Xdiv_{i_\mu}C_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_l}\phi_{u+1}$$
$$+\sum_{i=1}^{b_l}\sum_{i_y\in I_2^{T_i}}(-\sigma(i_y)+\tau(i_y))Xdiv_{i_1}\dots X\hat{d}iv_{i_y}\dots Xdiv_{i_\mu}$$
$$\nabla_{T_i}^{i_y}C_g^{l,i_1\dots\hat{i}_y\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_y}\phi_{u+1}$$
$$+\sum_{(i_k,i_l)\in I_1^{2,*}}\sum_{S=1}^{\sigma-1}Xdiv_{i_1}\dots X\hat{d}iv_{i_k}\dots X\hat{d}iv_{i_l}\dots Xdiv_{i_\mu}Xdiv_{i_z}$$
$$\tilde{C}_g^{l,i_1\dots,\hat{i}_k\dots,i_\mu,i_z|S}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_l}\phi_{u+1}\}$$
$$-\sum_{l\in L}a_l\sum_{i=1}^{b_l}\sum_{(i_k,i_l)\in(I_2^{T_i})^{2,def}}Xdiv_{i_1}\dots X\hat{d}iv_{i_k}\dots X\hat{d}iv_{i_l}\dots Xdiv_{i_a}$$
$$[C_g^{l,i_1\dots i_a,i_*|T_i}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)g^{i_ki_l}]\nabla_{i_*}\phi_{u+1}$$
$$-\sum_{l\in L}a_l\sum_{i=1}^{b_l}\sum_{i_k\in I_1^{T_i},i_l\in I_2^{T_i}}Xdiv_{i_1}\dots X\hat{d}iv_{i_k}\dots X\hat{d}iv_{i_l}\dots Xdiv_{i_a}$$
$$\nabla_{T_i}^{i_*}[C_g^{l,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)g^{i_ki_l}]\nabla_{i_*}\phi_{u+1}+$$
$$\sum_{j\in J}a_jC_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})+$$
$$\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_a}C_g^{h,i_1\dots i_a,i_*}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}=0, \tag{6.25}$$

modulo complete contractions of length $\geq\sigma+u+2$.

Finally, in the case where the selected factor is of the form $\nabla^{(A)}\Omega_h$ (in which case it is again unique), we derive:

$$\sum_{l\in L_\mu} a_l\{\sum_{i_h\in I_1}(\gamma+2A^\sharp-2\sigma_1-2\sigma_2)$$
$$Xdiv_{i_1}\dots X\hat{div}_{i_h}\dots Xdiv_{i_\mu}C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_h}\phi_{u+1}$$
$$-3\sum_{(i_k,i_l)\in I_1^{2,*}} Xdiv_{i_1}\dots X\hat{div}_{i_l}\dots Xdiv_{i_\mu}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_l}\phi_{u+1}$$
$$+\sum_{i_y\in I_2}(-\sigma(i_y)+\tau(i_y))Xdiv_{i_1}\dots X\hat{div}_{i_y}\dots Xdiv_{i_\mu}$$
$$[\nabla^{i_y}_{sel}C^{l,i_1\dots\hat{i}_y\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_y}\phi_{u+1}]+$$
$$\sum_{(i_k,i_l)\in I_1^{2,*}}\sum_{S=1}^{\sigma-1} Xdiv_{i_1}\dots X\hat{div}_{i_k}\dots X\hat{div}_{i_l}\dots Xdiv_{i_\mu}Xdiv_{i_z}$$
$$\tilde{C}^{l,i_1\dots,\hat{i}_k\dots,i_\mu,i_z|S}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_l}\phi_{u+1}\}$$
$$-\sum_{l\in L} a_l\sum_{(i_k,i_l)\in I_2^{2,def}} Xdiv_{i_1}\dots X\hat{div}_{i_k}\dots X\hat{div}_{i_l}\dots Xdiv_{i_a}$$
$$[C^{l,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)g^{i_ki_l}]\nabla_{i_*}\phi_{u+1}$$
$$-\sum_{l\in L} a_l\sum_{i_k\in I_1,i_l\in I_2} Xdiv_{i_1}\dots X\hat{div}_{i_k}\dots X\hat{div}_{i_l}\dots Xdiv_{i_a}$$
$$\nabla^{i_*}_{sel}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)g^{i_ki_l}]\nabla_{i_*}\phi_{u+1}+$$
$$\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})+$$
$$\sum_{h\in H} a_h Xdiv_{i_1}\dots Xdiv_{i_a}C^{h,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}=0, \tag{6.26}$$

modulo complete contractions of length $\ge \sigma+u+2$.

Now, in order to motivate our next section, let us briefly illustrate how the above three equations *are not* in the form required by Lemma 1.3 (in case A). There are two important defficiencies of the three equations above that we wish to highlight: Firstly, observe that in all equations above the sublinear combinations indexed in $\sum_{(i_k,i_l)\in I_2^{2,def}}$ have rank $(\mu-2)$. The desired conclusion of Lemma 1.3 requires the minimum rank of the tensor fields to be $\mu-1$. Secondly, the coefficient $(\gamma+(2\nu^\sharp+1)+\frac{\nu}{\nu+1}-2(\sigma_1+\sigma_2-1))$ (and also the coefficients in the fist lines of (6.25), (6.26)) *depends* on the number ν of derivatives on the selected factor in each individual tensor field in the second line in (6.24) (and similarly for the other two equations). This is not the case in the conclusion of Lemma 1.3 where all tensor fields of rank $\mu-1$ must be multiplied by a *universal* constant (either 1 or $\binom{\alpha}{2}$).

Thus, we observe that we *can not* derive our Lemma 1.3 merely by repeating the analysis of equation $Image^{1,+}_{\phi_{u+1}}[L_g]$ that we performed to derive the Lemmas

3.1 and 3.2 in [8]. The second ingredient in the proof of Lemma 1.3 will be an analysis of the equation $Image^{1,\beta}_{\phi_{u+1}}[L_g] = 0$ (see Definition 2.3 and equation (2.9) in [8]), coupled with a formal operation $Soph\{\dots\}$ which will then *convert* the terms in that equation into the type of terms that are required in Lemma 1.3.

7 The second part of the "grand conclusion": A study of $Image^{1,\beta}_{\phi_{u+1}}[L_g] = 0$.

7.1 Basic calculations in the equation $Image^{1,\beta}_{\phi_{u+1}}[L_g] = 0$, and first steps in its analysis.

We recall from definition 2.3 that $Image^{1,\beta}_{\phi_{u+1}}[L_g]$ consists of the complete contractions in $Image^{1}_{\phi_{u+1}}[L_g]$ that have one internal contraction, and also either have length $\sigma + u$ and a factor $\nabla^{(A)}\phi_{u+1}$, $A \geq 2$, or have length $\sigma + u + 1$ and a factor $\nabla\phi_{u+1}$. *Recall that the selected factor(s), the crucial factor(s) etc. are completely irrelevant in the context of* $Image^{1,\beta}_{\phi_{u+1}}[L_g]$. We recall that we have denoted by $Image^{1,\beta,\sigma+u}_{\phi_{u+1}}[L_g]$ the sublinear combination of contractions with $\sigma + u$ factors in $Image^{1,\beta}_{\phi_{u+1}}[L_g]$, and by $Image^{1,\beta,\sigma+u+1}_{\phi_{u+1}}[L_g]$ the sublinear combination of contractions with $\sigma + u + 1$ factors in $Image^{1,\beta}_{\phi_{u+1}}[L_g]$.

We recall that in (2.9) we showed that modulo complete contractions with at least $\sigma + u + 2$ factors:

$$Image^{1,\beta,\sigma+u}_{\phi_{u+1}}[L_g]+Image^{1,\beta,\sigma+u+1}_{\phi_{u+1}}[L_g]+\sum_{z\in Z} a_z C^z_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}) = 0. \tag{7.1}$$

(We recall that $\sum_{z\in Z}\dots$ stands for a *generic* linear combination of contractions with $\sigma + u + 1$, one of which is in the from $\nabla^{(B)}\phi_{u+1}$, $B \geq 2$).

Now, in the rest of this section we will study the LHS of the equation (7.1) and repeatedly apply the inductive assumption of Corollary 1 in [6] to it in many different guises, in order to derive a new local equation which will help us in deriving Lemma 1.3. In particular, the new local equation combined with (6.24), (6.25), (6.26) will give us our "grand conclusion".

Firstly, we focus of the complete contractions of length $\sigma+u$ in $Image^{1,\beta}_{\phi_{u+1}}[L_g]$.

We initially seek to understand *how* the complete contractions in $Image^{1,\beta}_{\phi_{u+1}}[L_g]$ can arise. For the purposes of this subsection, when we study the transformation law of each factor of the form $S_*\nabla^{(\nu)}R_{ijkl}$, we will be treating each such factor as a sum of factors in the form $\nabla^{(m)}R_{ijkl}$. With this convention we immediately see that the complete contractions of length $\sigma + u$ in $Image^{1,\beta}_{\phi_{u+1}}[L_g]$ can only arise by replacing a factor $\nabla^{(m)}R_{ijkl}$ by an expression $\nabla^{(m)}[\nabla^{(2)}_{cd}\phi_{u+1} \otimes g_{ab}]$, *provided the indices* $_{a}, _{b}$ *both contract against the same factor.*

We then recall the operation Sub_ω (from the Appendix in [3]) with which we will act on the complete contractions in $Image^{1,\beta}_{\phi_{u+1}}[L_g]$:

Definition 7.1 *If a complete contraction $C_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ has its internal contraction in a factor of the form $\nabla^{(p)}_{r_1\dots r_p}Ric_{ij}$, then $Sub_\omega[C_g]$ stands for the complete contraction that arises from C_g by replacing the factor $\nabla^{(p)}_{r_1\dots r_p}Ric_{ij}$ by a factor $-\nabla^{(p+2)}_{r_1\dots r_p ij}\omega$. If the internal contraction involves at least one derivative index (hence it is in the form $(\nabla^a,{}_a)$), then $Sub_\omega[C_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ stands for the complete contraction that arises from $C_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ by replacing the expression $(\nabla^a,{}_a)$ by an expression $(\nabla^a\omega,{}_a)$.*[101] *The above extends to linear combinations of complete contractions and also to tensor fields.*

Hence, by applying the last Lemma in the Appendix of [3] to the equation (7.1), we derive that:

$$Sub_\omega\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}+\sum_{t\in T}a_tC^t_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1},\omega)=0, \tag{7.2}$$

modulo complete contractions of length $\geq\sigma+u+3$. Here $\sum_{t\in T}a_tC^t_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\omega)$ stands for a generic linear combination of complete contractions with either length $\sigma+u+2$ and a factor $\nabla^{(A)}\omega, A\geq 2$, or a factor $\nabla^{(A)}\phi_{u+1}$, $A\geq 2$, *or* with length $\sigma+u+1$ and a factor $\nabla^{(A)}\phi_{u+1}, A\geq 2$ *and* a factor $\nabla^{(B)}\omega, B\geq 2$. This equation follows from the last Lemma in the Appendix of [3], and the definition of the various terms in (7.1).

A brief analysis of equation (7.2): We observe that by definition $Sub_\omega\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}$ consists of three main sublinear combinations,[102] depending on the total number of factors that a given complete contraction in $Sub_\omega\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}$ contains: A given complete contraction in $Sub_\omega\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}$ may contain $\sigma+u,\sigma+u+1$ or $\sigma+u+2$ factors in total. Accordingly, we denote the corresponding sublinear combinations by $Sub^{\sigma+u}_\omega\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}$, $Sub^{\sigma+u+1}_\omega\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}$, $Sub^{\sigma+u+2}_\omega\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}$. By definition, the complete contractions in $Sub^{\sigma+u}_\omega\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}$ will have two factors $\nabla^{(A)}\phi_{u+1},\nabla^{(B)}\omega$, $A,B\geq 2$. The complete contractions in $Sub^{\sigma+u+1}_\omega\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}$ will have two factors, either in the form $\nabla^{(A)}\phi_{u+1},\nabla\omega$, $A\geq 2$ *or* in the form $\nabla\phi_{u+1},\nabla^{(B)}\omega$, $B\geq 2$. Finally, the complete contractions in $Sub^{\sigma+u+2}_\omega\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}$ will have two factors in the form $\nabla\phi_{u+1},\nabla\omega$.

Thus, (7.2) can be re-expressed in the form:

[101] In other words the derivative index ∇^a is erased, and a new factor $\nabla\omega$ is introduced, which is then contracted against the index $_a$ ($_a$ is the index that originally contracted against ∇^a).

[102] Plus junk terms of greater length.

$$Sub_\omega^{\sigma+u}\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}+Sub_\omega^{\sigma+u+1}\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}+ \\ Sub_\omega^{\sigma+u+2}\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}+\sum_{t\in T}a_tC^t_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1},\omega)=0. \tag{7.3}$$

(modulo complete contractions of length $\geq \sigma+u+3$). For future reference, we will also divide the index set T into subsets $T^{\sigma+u+1}, T^{\sigma+u+2}$, according to the number of factors that a given complete contraction $C^t_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1},\omega)$ contains.

We first seek to understand the sublinear combination $Sub_\omega^{\sigma+u}\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}$ in $Sub_\omega\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}$ which consists of complete contractions of length $\sigma+u$. We start by studying how $Sub_\omega\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}$ arises from the equation $L_g=0$ and we will then prove two important equations regarding this sublinear combination, namely equations (7.9) and (7.12). *Important Note:* For future reference, we note here that all the discussion until (7.9) and (7.12), and also the proofs of these equations also hold *without* the assumptions that $L^*_\mu\bigcup L^+_\mu\bigcup L''_+=\emptyset$. This will be used in the proof of Lemmas 3.3, 3.4 from [6]. Furthermore, the assumptions $L''_+=\emptyset$ will not be used until after equation (7.36).

How does the sublinear combination $Sub_\omega^{\sigma+u}\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}$ arise?

For each tensor field $C^{l,i_1\dots i_a}_g$ and each complete contraction C^j_g in L_g, we look for the pairs of non-anti-symmetric internal indices $({}_a,{}_b),({}_c,{}_d)$ in two different factors $\nabla^{(m)}R_{ijkl}$ (recall that we are treating the factors $S_*\nabla^{(\nu)}R_{ijkl}$ as sums of tensors in the form $\nabla^{(m)}R_{ijkl}$) for which ${}_a,{}_c$ and ${}_b,{}_d$ are contracting against each other.

In each tensor field $C^{l,i_1\dots i_a}_g$ or complete contraction C^j_g, we denote the set of those pairs by INT^2_l, INT^2_j, respectively. For each such pair $[({}_a,{}_b),({}_c,{}_d)]\in INT^2_l$ or $[({}_a,{}_b),({}_c,{}_d)]\in INT^2_j$, we denote by $Rep^1[C^{l,i_1,\dots i_a}_g]$, $Rep^1[C^j_g]$ the complete contraction or tensor field that arises by replacing the first factor $\nabla^{(m)}R_{ijkl}$ by the linear expression $(+-)\nabla^{(m+2)}\phi\otimes g_{ab}$ on the right hand side of (2.1) and then applying Sub_ω (here, of course, the $+/-$ sign comes from (2.1)). We also denote by $Rep^2[C^j_g]$, $Rep^2[C^{l,i_1,\dots,i_a}_g]$ the tensor field (or complete contraction) that arises by replacing the second factor $\nabla^{(m')}R_{ijkl}$ by the linear expression $(+-)\nabla^{(m'+2)}\phi_{u+1}\otimes g_{cd}$ on the right hand side of (2.1) and then applying Sub_ω.

We define:

$$Rep[C^{l,i_1,\dots i_a}_g]=\sum_{[(a,b),(c,d)]\in INT^2_l}\{Rep^1_{[(a,b),(c,d)]}[C^{l,i_1,\dots,i_a}_g] \\ +Rep^2_{[(a,b),(c,d)]}[C^{l,i_1,\dots,i_a}_g]\} \tag{7.4}$$

and

$$Rep[C^j_g] = \sum_{[(a,b),(c,d)]\in INT^2_l} \{Rep^1_{[(a,b),(c,d)]}[C^j_g] + Rep^2_{[(a,b),(c,d)]}[C^j_g]\}. \tag{7.5}$$

We straightforwardly observe that:

$$\begin{aligned} &Sub^{\sigma+u}_\omega\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\} = \sum_{j\in J} a_j Rep[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] \\ &\sum_{l\in L} a_l Xdiv_{i_1}\dots Xdiv_{i_a} Rep[C^{l,i_1,\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]. \end{aligned} \tag{7.6}$$

Moreover, by (7.2) we have that:

$$Sub^{\sigma+u}_\omega\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\} = 0, \tag{7.7}$$

modulo complete contractions of length $\geq \sigma+u+1$.

Before we move on to examine the other sublinear combinations of $Sub^{\sigma+u}_\omega\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}$, we want to somehow apply Corollary 1 in [6] to (7.7). We introduce some notational conventions:

Definition 7.2 *We will denote by*

$$\sum_{j\in J^{\sigma+u}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)$$

a generic linear combination of complete contractions with length $\sigma+u$ and two factors $\nabla^{(A)}\phi_{u+1}, \nabla^{(B)}\omega$ and at least one factor $\nabla\phi_f$, $f\in Def(\vec{\kappa}_{simp})$ contracting against a derivative index in a factor $\nabla^{(m)}R_{ijkl}$ or contracting against one of the first $A-2$ indices in a factor $\nabla^{(A)}\phi_{u+1}$ or $\nabla^{(A)}\omega$.

We denote by

$$\sum_{u\in U_1} a_u C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)$$

($a\geq\mu$) a generic linear combination of acceptable tensor fields with length $\sigma+u+1$, a factor $\nabla\phi_{u+1}$ (which we treat as a factor $\nabla\phi$)[103] *and a factor $\nabla^{(A)}\omega$, $A\geq 2$ (which we treat as a factor $\nabla^{(A)}\Omega_{p+1}$), with the additional feature that the u-simple character of $C^{u,i_1\dots,i_a}_g$ arises from $\vec{\kappa}_{simp}$ by replacing one factor $T=\nabla^{(m)}R_{ijkl}$ or $T=S_*\nabla^{(\nu)}R_{ijkl}$ by a factor $T'=\nabla^{(\nu+2)}\Omega_{p+1}$, where all the factors $\nabla\phi_h$ that contracted against T now contract against T'; moreover if $a=\mu$ and the factor $\nabla^{(A)}\omega$ has $A=2$ and is contracting against a factor $\nabla\phi_h$ then we additionnaly require that it does not contain free indices.*

We denote by

$$\sum_{u\in U_2} a_u C^{u,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)$$

[103] In particular this factor does not contain a free index.

a generic linear combination of acceptable tensor fields with length $\sigma+u+1$, a factor $\nabla\omega$ (which we treat as a factor $\nabla\phi$)[104] *and a factor $\nabla^{(A)}\phi_{u+1}$, $A\ge 2$ (which we treat as a factor $\nabla^{(A)}\Omega_{p+1}$), with the restriction that the u-simple character of $C^{u,i_1\dots i_a}_g$ arises from $\vec{\kappa}_{simp}$ by replacing one factor $T=\nabla^{(m)}R_{ijkl}$ or $T=S_*\nabla^{(\nu)}R_{ijkl}$ by a factor $T'=\nabla^{(A)}\Omega_{p+1}$ where all the factors $\nabla\phi_h$ that contracted against T now contract against T'; moreover if $a=\mu$ and the factor $\nabla^{(A)}\phi_{u+1}$ has $A=2$ and is contracting against a factor $\nabla\phi_h$ then we additionnaly require that it does not contain free indices.*

In addition, we will denote by

$$\sum_{u\in U_1^\sharp} a_u C^{u,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega),$$

$$\sum_{u\in U_2^\sharp} a_u C^{u,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)$$

generic linear combinations as above with one un-acceptable factor of the form $\nabla\Omega_h$, $h\le p$, (with only one derivative) contracting against a factor $\nabla\phi_{u+1}$ or $\nabla\omega$, respectively.

We will denote by

$$\sum_{j\in J^{\sigma+u+1}} a_j C^{j}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)$$

a generic linear combination of complete contractions with length $\sigma+u+1$ and two factors either $\nabla^{(A)}\phi_{u+1},\nabla\omega$ or $\nabla^{(A)}\omega,\nabla\phi_{u+1}$ ($A\ge 2$) and at least one factor $\nabla\phi_f, f\in Def(\vec{\kappa}_{simp})$[105] *contracting against a derivative index in a factor $\nabla^{(m)}R_{ijkl}$ or contracting against one of the first $A-2$ indices in a factor $\nabla^{(A)}\phi_{u+1}$ or $\nabla^{(A)}\omega$.*

For future reference, we will also put down some definitions regarding complete contractions of length $\sigma+u+2$ in $Sub_\omega\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}$ (with a factor $\nabla\phi_{u+1}$ and a factor $\nabla\omega$): We will denote by

$$\sum_{m\in M} a_m C^{m,i_1\dots i_{a+2}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{a+1}}\phi_{u+1}\nabla_{i_{a+2}}\omega$$

(where $a\ge\mu$) a generic linear combination of acceptable tensor fields of length $\sigma+u+2$ with a u-simple character $\vec{\kappa}_{simp}$ (this u-simple character only encodes information on the factors $\nabla\phi_h, h\le u$).

We also denote by

$$\sum_{m\in M^\sharp} a_m C^{m,i_1\dots i_{a+2}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{a+1}}\phi_{u+1}\nabla_{i_{a+2}}\omega$$

[104] In particular this factor does not contain a free index.

[105] Recall that $Def(\vec{\kappa}_{simp})$ stands for the set of numbers o for which $\nabla\tilde{\phi}_o$ is contracting against the index $_i$ in some factor $S_*\nabla^{(\nu)}R_{ijkl}$.

a generic linear combination of tensor fields as above, with one un-acceptable factor of the form $\nabla_{i_b}\Omega_f$ *where* $_{i_b}$ *is a free index and in fact* $b = a+1$ *or* $b = a+2$.

We will denote by

$$\sum_{j\in J^{\sigma+u+2}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)$$

a generic linear combination of complete contractions with length $\sigma+u+2$ *and two factors* $\nabla\phi_{u+1},\nabla\omega$ *and at least one factor* $\nabla\phi_f, f\in Def(\vec{\kappa}_{simp})$ *contracting against a derivative index in a factor* $\nabla^{(m)}R_{ijkl}$.

Finally, we will denote by

$$\sum_{w\in W} a_w C^w_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)$$

a generic linear combination of complete contractions that have length $\geq \sigma+u+1$ *and two factors* $\nabla^{(A)}\phi_{u+1},\nabla^{(B)}\omega$ *with* $A,B\geq 2$.[106]

Now, return to (7.6). We wish to analyze the terms in this equation. We firstly observe that:

$$\sum_{j\in J} a_j Rep[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] = \sum_{j\in J^{\sigma+u}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega). \tag{7.8}$$

Now, we claim that we can write out:

$$\begin{aligned}
&\sum_{l\in L} a_l Xdiv_{i_1}\dots Xdiv_{i_a} Rep[C^{l,i_1,\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)] = \\
&\sum_{u\in U_1} a_u Xdiv_{i_1}\dots Xdiv_{i_a} C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+ \\
&\sum_{u\in U_2} a_u Xdiv_{i_1}\dots Xdiv_{i_a} C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+ \\
&\sum_{j\in J^{\sigma+u}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+ \\
&\sum_{w\in W} a_w C^w_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega);
\end{aligned} \tag{7.9}$$

(this holds perfectly, *not* modulo complete contractions of greater length).

If we can prove this, we can then replace into (7.2) to derive an equation:

[106]Notice that for length $\sigma+u+1$, the sublinear combination $\sum_{w\in W}\dots$ corresponds exactly to the generic linear combination $\sum_{t\in T_1}\dots$ in (7.3).

$$Sub^{\sigma+u}_{\omega}[L_g] = \sum_{u\in U_1} a_u Xdiv_{i_1}\dots Xdiv_{i_a} C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+$$
$$\sum_{u\in U_2} a_u Xdiv_{i_1}\dots Xdiv_{i_a} C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+$$
$$\sum_{j\in J^{\sigma+u}_1} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+$$
$$\sum_{w\in W} a_w C^w_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega), \tag{7.10}$$

(the above holds perfectly), where the tensor fields indexed in $J^{\sigma+u}_1$ are of the same general form as the tensor fields we generically index in $J^{\sigma+u}$. In other words, provided we can show (7.9) we derive that modulo complete contractions of length $\geq \sigma+u+1$:

$$\sum_{j\in J^{\sigma+u}_1} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega) = 0, \tag{7.11}$$

(modulo length $\geq \sigma+u+1$).

Then, using the above we will show that we can write:

$$\begin{aligned}&\sum_{j\in J^{\sigma+u}_1} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega) =\\ &\sum_{j\in J^{\sigma+u+1}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)\\ &+\sum_{w\in W} a_w C^w_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega).\end{aligned} \tag{7.12}$$

(The above holds exactly, not modulo longer complete contractions). Moreover, the left hand side is not generic notation, but stands for the same sublinear combination in (7.9). The right hand side consists of generic linear combinations as defined in definition 7.2.

7.2 Further steps in the analysis of $Image^{1,\beta}_{\phi_{u+1}}[L_g] = 0$: Proof of (7.9) and (7.12):

We first show (7.9). We will prove this equation using the inductive assumption on Proposition 1.1 and the usual inductive argument: We will break up (7.7) into sublinear combinations that have the same u-weak character (here the weak character also takes into account the two new factors $\Omega_{p+1}=\phi_{u+1},\Omega_{p+2}=\omega$), and then inductively apply Corollary 1 from [6] and at each step we convert the $\nabla \upsilon$'s in $Xdiv$'s.

Specifically: For each $C_g^{l,i_1\dots i_a}$, C_g^j we divide their curvature factors ($\nabla^{(m)}R_{ijkl}, S_*\nabla^{(\nu)}R_{ijkl}$) into *categories*: $K_1,\dots,K_d$. We decide that two curvature factors T,T' in two complete contractions or tensor fields in (7.9) belong to the same category if they are contracting against the same factors $\nabla\phi_h$. We also decide that the curvature factors that do not contract against any factors $\nabla\phi_h$ belong to the last category K_d.

We then recall that each tensor field and each complete contraction in (7.9) has arisen by replacing one curvature factor $\nabla^{(v)}R_{ijkl}$ by $\nabla^{(v)}[\nabla^{(2)}_{ab}\phi_{u+1}]$ and one other curvature factor $\nabla^{(c)}R_{i'j'k'l'}$ by $\nabla^{(c)}[\nabla^{(2)}_{a'b'}\omega]$. We then index the tensor fields and complete contractions in (7.9) in the sets $L^{\alpha,\beta}, J^{\alpha,\beta}$ ($1\le\alpha,\beta\le d$) according to the rule that a tensor field or complete contraction in (7.9) belongs to $L^{\alpha,\beta}, J^{\alpha,\beta}$ if and only if it has arisen by replacing a curvature factor that belongs to the category K_α by $\nabla^{(c+2)}_{ij}\phi_{u+1}$ and a curvature factor that belongs to K_β by $\nabla^{(v+2)}_{ij}\omega$.

We then see that (7.9) can be re-written in the form:

$$\sum_{1\le\alpha,\beta\le d}\{\sum_{L^{\alpha,\beta}} a_l Xdiv_{i_1}\dots Xdiv_{i_a}C_g^{l,i_1\dots i_a}+\sum_{j\in J^{\alpha,\beta}} a_j C_g^j\}=0, \tag{7.13}$$

modulo complete contractions of length $\ge\sigma+u+1$. Then, since the above must hold formally and hence sublinear combinations with different weak characters must vanish separately, we derive that for each pair $1\le\alpha,\beta\le d$:

$$\sum_{l\in L^{\alpha,\beta}} a_l Xdiv_{i_1}\dots Xdiv_{i_a}C_g^{l,i_1\dots i_a}+\sum_{j\in J^{\alpha,\beta}} a_j C_g^j=0, \tag{7.14}$$

(modulo complete contractions of length $\ge\sigma+u+1$).

We will then show our claim (7.9) for each of the index sets $L^{\alpha,\beta}$ separately. We distinguish three cases: Either both the categories K_a,K_b represent a "genuine" factor $\nabla^{(m)}R_{ijkl}$ *or* one of them represents such a "genuine" factor and the other represents a factor $S_*\nabla^{(\nu)}R_{ijkl}$ *or* both represent factors $S_*\nabla^{(\nu)}R_{ijkl}$. We denote these three cases by $(i),(ii),(iii)$ respectively.

Proof of (7.9) in case (i): We observe that all tensor fields indexed in $L^{\alpha,\beta}$ have the *same* u-simple character, say $\tilde{\kappa}_{simp}$ (we are treating the factors $\nabla^{(u)}\phi_{u+1},\nabla^{(y)}\omega$ as functions $\Omega_{p+1},\Omega_{p+2}$). Moreover, each of the tensor fields indexed in $L^{\alpha,\beta}$ has the property that any factors $\nabla\phi_h$ that are contracting against the factors $\nabla^{(u)}\phi_{u+1},\nabla^{(y)}\omega$ will be contracting against one of the left-most $u-2$ or $y-2$ indices. This follows by the definition above. We also note that all the complete contractions indexed in $J^{\alpha,\beta}$ are simply subsequent to $\tilde{\kappa}_{simp}$. We denote by τ the minimum rank of the tensor fields appearing in $L^{\alpha,\beta}$ (by hypothesis $\tau\ge\mu$). We index those tensor fields in the set $L^{\alpha,\beta|\tau}$. Therefore, (after using the Eraser, defined in the Appendix of [3], if necessary) we may apply the inductive assumption of Corollary 1 in [6][107] to derive that there

[107]Observe that (7.14) falls under the inductive assumption of Corollary 1 in [6], because the

exists a linear combination of acceptable tensor fields, $\sum_{h\in H^{\alpha,\beta}} a_h C_g^{h,i_1\dots i_{\tau+1}}$ of u-simple character $\tilde{\kappa}_{simp}$ and with the additional restriction that any factor $\nabla\phi_h$ that is contracting against $\nabla^{(u)}\phi_{u+1}$ or $\nabla^{(y)}\omega$ is contracting against one of the $u-2$ or $y-2$ leftmost indices and moreover with rank $\tau+1$ so that:

$$\sum_{l\in L^{\alpha,\beta|\tau}} a_l C_g^{l,i_1\dots i_\tau}\nabla_{i_1}\upsilon\dots\nabla_{i_\tau}\upsilon - \sum_{h\in H^{\alpha,\beta}} a_h Xdiv_{i_{\tau+1}} C_g^{h,i_1\dots i_{\tau+1}}\nabla_{i_1}\upsilon\dots\nabla_{i_\tau}\upsilon = \sum_{j\in J^{\sigma+u}} a_j C_g^{j,i_1\dots i_\tau}\nabla_{i_1}\upsilon\dots\nabla_{i_\tau}\upsilon, \tag{7.15}$$

modulo complete contractions of length $\geq \sigma+u+1+\tau$. Here the tensor fields indexed in $J^{\sigma+u}$ have the same properties as the contractions indexed in $J^{\sigma+u}$.

In fact since the above holds formally, by just paying attention to the correction terms of greater length that arise in (7.15) we derive:

$$\begin{aligned}&\sum_{l\in L^{\alpha,\beta|\tau}} a_l C_g^{l,i_1\dots i_\tau}\nabla_{i_1}\upsilon\dots\nabla_{i_\tau}\upsilon - \sum_{h\in H^{\alpha,\beta}} a_h Xdiv_{i_{\tau+1}} C_g^{h,i_1\dots i_{\tau+1}}\nabla_{i_1}\upsilon\dots\nabla_{i_\tau}\upsilon \\ &= \sum_{u\in U_1} a_u C_g^{u,i_1\dots,i_\tau}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)\nabla_{i_1}\upsilon\dots\nabla_{i_\tau}\upsilon+ \\ &\sum_{u\in U_2} a_u C_g^{u,i_1\dots,i_\tau}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)\nabla_{i_1}\upsilon\dots\nabla_{i_\tau}\upsilon+ \\ &\sum_{j\in J^{\sigma+u}} a_j C_g^{j,i_1\dots i_\tau}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)\nabla_{i_1}\upsilon\dots\nabla_{i_\tau}\upsilon+ \\ &\sum_{w\in W} a_w C_g^{w,i_1\dots i_\tau}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)\nabla_{i_1}\upsilon\dots\nabla_{i_\tau}\upsilon,\end{aligned} \tag{7.16}$$

modulo complete contractions of length $\geq \sigma+u+2+\tau$. Here the tensor fields indexed in W have the same properties as the complete contractions indexed in W: They have length $\sigma+u+1$ but also have two factors $\nabla^{(A)}\phi_{u+1},\nabla^{(B)}\omega$, $A,B\geq 2$.

Therefore, making the $\nabla\upsilon$'s into $Xdiv$'s[108] we derive an equation:

tensor fields there have length $\sigma+u$ and $p+2$ factors $\nabla^{(y)}\Omega_h$. Moreover observe that here is no danger of falling under a "forbidden" case of Corollary 1 in [6], since all the μ-tensor fields in (1.6) have no special free indices, thus there will be no special free indices among the tensor fields of minimum rank in (7.14).

[108](We are using the last Lemma from the Appendix in [3]).

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_\tau}\sum_{l\in L^{\alpha,\beta|\tau}} a_l C^{l,i_1\dots i_\tau}_g - \sum_{h\in H^{\alpha,\beta}} a_h Xdiv_{i_1}\dots Xdiv_{i_\tau} Xdiv_{i_{\tau+1}}\\
&C^{h,i_1\dots i_{\tau+1}}_g = Xdiv_{i_1}\dots Xdiv_{i_\tau}\sum_{u\in U_1} a_u C^{u,i_1\dots,i_\tau}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+\\
&\sum_{u\in U_2} a_u Xdiv_{i_1}\dots Xdiv_{i_\tau} C^{u,i_1\dots,i_\tau}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+\\
&\sum_{j\in J^{\sigma+u}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+\\
&\sum_{w\in W} a_w C^w_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega).
\end{aligned}
\tag{7.17}
$$

Thus, substituting into (7.14) and inductively repeating this step,[109] we derive our claim in this first case.

Proof of (7.9) in the case (ii): Now, in the second case we assume with no loss of generality that K_α corresponds to a factor $\nabla^{(m)}R_{ijkl}$ in $\vec{\kappa}_{simp}$ and K_β corresponds to a factor $S_*\nabla^{(\nu)}R_{ijkl}\nabla^i\tilde{\phi}_f$ in $\vec{\kappa}_{simp}$. We again observe that all the tensor fields $C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1},\omega)$ have the same u-simple character $\vec{\kappa}'_{simp}$ and any factor $\nabla\phi_h$ that is contracting against the factor $\nabla^{(p)}\phi_{u+1}$ in any tensor field $C^{l,i_1\dots i_a}_g$ in (7.14) must be contracting against one of the first $p-2$ indices there.

Then, for each vector field $C^{l,i_1\dots i_a}_g, l\in L^{\alpha,\beta}$ we inquire whether the factor $\nabla^{(p)}_{r_1\dots r_p}\omega$ (for which ${}_{r_{p-1}}$ is contracting against the factor $\nabla\tilde{\phi}_f$) has $p>2$ or $p=2$. In the first case, we straightforwardly observe that we can write:

$$
\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega) =\\
&\sum_{j\in J^{\sigma+u}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+\\
&\sum_{u\in U_2} a_u Xdiv_{i_1}\dots Xdiv_{i_a} C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)\\
&+\sum_{w\in W} a_w C^w_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega);
\end{aligned}
\tag{7.18}
$$

(the above holds perfectly, not modulo terms of greater length). This just follows by applying the curvature identity to ${}_{r_{p-2}},{}_{r_{p-1}}$ in $\nabla^{(p)}_{r_1\dots r_p}\omega$. Thus, we may assume with no loss of generality that all the tensor fields indexed in $L^{\alpha,\beta}$ have a factor $\nabla^{(2)}\omega$.

[109] At the very last step of this inductive argument, we may have to apply the "weak substitute" of Proposition 1.1, from the Appendix of [6]. In that case our result will follow since in that case the minimum rank among those terms will be $>\mu$.

By the same argument, we may also assume that all complete contractions indexed in $J^{\alpha,\beta}$ either have a factor $\nabla^{(p)}\omega, p \geq 3$ or a factor $\nabla^{(2)}\omega$ but one of the factors $\nabla\phi_h$, $h \in Def(\vec{\kappa}_{simp})$ is contracting against a derivative index in some factor $\nabla^{(m)}R_{ijkl}$. Accordingly, we break up $J^{\alpha,\beta}$ into $J^{\alpha,\beta,I}$, $J^{\alpha,\beta,II}$.

Therefore, picking out the sublinear combination in (7.6) with a factor $\nabla^{(2)}\omega_1$, we derive the equation:

$$\begin{aligned}
&\sum_{l\in L^{\alpha,\beta}} a_l X_* div_{i_1}\dots X_* div_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+ \\
&\sum_{j\in J^{\alpha,\beta,II}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)=0,
\end{aligned} \tag{7.19}$$

where X_*div_i stands for the sublinear combination in $Xdiv_i$ where ∇_i is not allowed to hit the factor $\nabla^{(2)}\omega$. Furthermore, we may assume that for each of the tensor fields above, the factor $\nabla^{(2)}\omega$ does not contain a free index, yet the rank of all the tensor fields is $\geq \mu$: This assumption can be made with no loss of generality since if a tensor field in (7.19) has $a = \mu$ then it will not contain a free index in $\nabla^{(2)}\omega$ by definition, while if $a > \mu$, we may just *neglect* the X_*div_i, where ${}_i$ is the (unique) free index belonging to $\nabla^{(2)}\omega_1$–the resulting iterated $Xdiv$ will still have rank $\geq \mu$.

Then, by using the eraser onto the factor $\nabla\phi_h$ that contracts against $\nabla^{(2)}\omega$ and applying our inductive assumption of Corollary 2 from [6][110] (or Lemma 4.7 if $\sigma = 3$), we derive that we can write:

$$\begin{aligned}
&\sum_{l\in L^{\alpha,\beta}} a_l Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)= \\
&\sum_{l\in L'^{\alpha,\beta}} a_l Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+ \\
&\sum_{j\in J^{\alpha,\beta,II}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega),
\end{aligned} \tag{7.20}$$

where the tensor fields indexed in $L'^{\alpha,\beta}$ have a factor $\nabla^{(3)}_{ijk}\omega\nabla^j\tilde{\phi}_f$. Also $\sum_{j\in J^{\alpha,\beta,II}} a_j \dots$ on the RHS is generic notation. Thus, using the identity $\nabla^{(3)}_{ijk}\omega\nabla^j\tilde{\phi}_f = \nabla^{(3)}_{jik}\omega\nabla^j\tilde{\phi}_f + R_{ijlk}\nabla^l\omega_1\nabla^k\tilde{\phi}_f$ we derive our claim.

Proof of (7.9) in case (iii): Finally, the last case, where both K_α, K_β are in the form $S_*\nabla^{(\nu)}R_{ijkl}$: As before, we consider any tensor field $C^{l,i_1\dots i_a}_g$ in (7.6) and we note that it must contain two expressions $\nabla^{(y)}_{r_1\dots r_y}\phi_{u+1}\nabla^{r_{y-1}}\tilde{\phi}_b$ and $\nabla^{(p)}_{t_1\dots t_p}\omega\nabla^{t_{p-1}}\tilde{\phi}_c$ with $y, p \geq 2$. Moreover all tensor fields have the same u-simple character which we denote by $\tilde{\kappa}_{simp}$. Furthermore the last index in both these

[110] Observe that since $\nabla\omega$ does not contain free indices we do not have to worry about the "forbidden cases".

factors is not contracting against a factor $\nabla\phi_h$. If either of the numbers y,p is >2, we then apply the curvature identity to derive that we can write:

$$\begin{aligned}
&Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)=\\
&\sum_{j\in J^{\sigma+u}}a_jC^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+\\
&\sum_{u\in U_1}a_uXdiv_{i_1}\dots Xdiv_{i_a}C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+\\
&\sum_{u\in U_2}a_uXdiv_{i_1}\dots Xdiv_{i_a}C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)\\
&+\sum_{w\in W}a_wC^w_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega).
\end{aligned}\tag{7.21}$$

Therefore, we may now prove our claim under the assumption that all tensor fields $C^{l,i_1\dots,i_a}_g$, $l\in L^{\alpha,\beta}$ have two factors $\nabla^{(2)}\phi_{u+1},\nabla^{(2)}\omega$, with the indices before last contracting against factors $\nabla\phi_h$, $h\in Def(\vec{\kappa}_{simp})$.

Analogously, we again divide $J^{\alpha,\beta}$ into two subsets. We say $j\in J^{\alpha,\beta,I}$ if at least one of the factors $\nabla^{(u)}\phi_{u+1}$ or $\nabla^{(y)}\omega$ has $u\geq 3$ or $y\geq 3$. We say $j\in J^{\alpha,\beta,II}$ if we have two factors $\nabla^{(2)}\phi_{u+1}$ and $\nabla^{(2)}\omega$. In this second case we see that by definition we must have at least one factor $\nabla\phi_h$, $h\in Def(\vec{\kappa}_{simp})$ contracting against a derivative index in some factor $\nabla^{(m)}R_{ijkl}$.

Now, picking out the tensor fields in (7.6) with two factors $\nabla^{(2)}\phi_{u+1},\nabla^{(2)}\omega$ we derive that:

$$\begin{aligned}
&\sum_{l\in L^{\alpha,\beta}}a_lX_*div_{i_1}\dots X_*div_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+\\
&\sum_{j\in J^{\alpha,\beta,II}}a_jC^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)=0,
\end{aligned}\tag{7.22}$$

where here X_*div_i means ∇_i is not allowed to hit the factor $\nabla^{(2)}\phi_{u+1}$ nor the factor $\nabla^{(2)}\omega$.

We then claim that we can write:

$$\begin{aligned}
&\sum_{l\in L^{\alpha,\beta}}a_lXdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)=\\
&\sum_{l\in L'^{\alpha,\beta}}a_lXdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+\\
&\sum_{j\in J^{\sigma+u}}a_jC^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega),
\end{aligned}\tag{7.23}$$

where each tensor field indexed in $L'^{\alpha,\beta}$ has a simple character $\tilde{\kappa}_{simp}$ and either an expression $\nabla^{(3)}_{ijk}\phi_{u+1}\nabla^j\tilde{\phi}_h$ or an expression $\nabla^{(3)}_{ijk}\omega\nabla^j\tilde{\phi}_h$ and has $a\geq\mu$. If we

can do this, then repeating the curvature identity as above we can derive our claim (7.9) in case (*iii*).

The proof of equation (7.23) is rather technical and the methods used there are not relevant to our further study of the sublinear combination $Image^{1,\beta}_{\phi_{u+1}}[L_g] = 0$. Thus, in order not to distract the reader from the main points of the argument, we will present the proof of (7.23) in the Mini-Appendix 8.4 below.

Proof of (7.12): (7.12) follows from (7.11) by the usual argument where we make the linearized complete contractions hold formally and then repeat the permutations to the non-linearized setting: We have that

$$\sum_{j\in J_1^{\sigma+u}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega) = 0,$$

modulo longer complete contractions, so the above holds formally at the linearized level, so repeating the permutations to the non-linearized level we get correction terms of the desired form. □

In conclusion, using (7.9) and (7.12), we can replace the first sublinear combination $Sub_\omega[\dots]$ in (7.3) to obtain a new equation:

$$\begin{aligned}
0 = &\sum_{u\in U_1} a_u Xdiv_{i_1}\dots Xdiv_{i_a} C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+\\
&\sum_{u\in U_2} a_u Xdiv_{i_1}\dots Xdiv_{i_a} C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+\\
&\sum_{j\in J^{\sigma+u}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+\\
&\sum_{w\in W} a_w C^w_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega) + Sub^{\sigma+u+1}_\omega\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}+\\
&Sub^{\sigma+u+2}_\omega\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\} + \sum_{t\in T^{\sigma+u+1}\bigcup T^{\sigma+u+2}} a_t C^t_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1},\omega).
\end{aligned} \tag{7.24}$$

In particular, since the minimum length of the complete contractions in the above is $\sigma+u+1$, if we denote by F_g the sublinear combination of terms with $\sigma+u+1$ factors with two factors $\nabla^{(A)}\phi_{u+1}, \nabla^{(B)}\omega$, then $lin\{F_g\}=0$ formally. Now, notice that the sublinear combination F_g is in fact:

$$\begin{aligned}
F_g = &\sum_{w\in W} a_w C^w_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+\\
&\sum_{t\in T^{\sigma+u+1}} a_t C^t_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1},\omega).
\end{aligned} \tag{7.25}$$

Then, using the fact that $lin\{F_g\}=0$ formally, we repeat the permutations that make the LHS of this equation zero to the *non-linear* setting, to derive:

$$F_g=\sum_{t\in T^{\sigma+u+2}} a_t C^t_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1},\omega). \tag{7.26}$$

(This equation holds perfectly, not "modulo longer terms"), using generic notation on the RHS. Therefore, plugging the above into (7.24), we derive a new equation:

$$\begin{aligned}
&\sum_{u\in U_1} a_u Xdiv_{i_1}\dots Xdiv_{i_a} C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+\\
&\sum_{u\in U_2} a_u Xdiv_{i_1}\dots Xdiv_{i_a} C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+\\
&\sum_{j\in J^{\sigma+u}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+Sub^{\sigma+u+1}_\omega\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}+\\
&Sub^{\sigma+u+2}_\omega\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}+\sum_{t\in T^{\sigma+u+2}} a_t C^t_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1},\omega)=0.
\end{aligned} \tag{7.27}$$

We will keep the equation (7.27) in mind. We now set out to study the sublinear combination $Sub^{\sigma+u+1}_\omega\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}$ in the above, which (we recall) consists of the complete contractions in $Sub_\omega\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}$ which have $\sigma+u+1$ factors.

7.3 A study of the sublinear combination $Sub^{\sigma+u+1}_\omega\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}$.

In order to understand *how* the sublinear combination $Sub^{\sigma+u+1}_\omega\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}$ arises from $L_g=0$, we must study certain *special patterns* of particular contractions among the different complete contractions in L_g:

We think of L_g as a linear combination of complete contractions (i.e. we momentarily forget its structure-that it contains a linear combination of X-divergences), and we also break the S_*-symmetrization in the factors $S_*\nabla^{(\nu)}R_{ijkl}-$ i.e. we treat those terms as sums of factors $\nabla^{(\nu)}R_{ijkl}$. Thus we write out:

$$L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)=\sum_{v\in V} a_v C^v_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u). \tag{7.28}$$

We then separately examine each complete contraction C^v_g in L_g and we consider all the *pairs of pairs* of indices, $[({}_a,{}_b),({}_c,{}_d)]$ where ${}_a,{}_b$ are two non-antisymmetric internal indices in a factor $\nabla^{(m)}R_{ijkl}$ and the two indices ${}_c,{}_d$ belong to some

other factor and ${}_c$ is a derivative index, so that ${}_a$ is contracting against ${}_c$ and ${}_b$ against ${}_d$.

We denote the set of such pairs in each complete contraction C^v_g in L_g by INT^2_v (or just INT^2 for short). Then, for each $[({}_a, {}_b), ({}_c, {}_d)] \in INT^2$, we denote by $Rep^1_{[(a,b),(c,d)]}[C^v_g]$ the complete contraction that formally arises from C^v_g by replacing the first factor $\nabla^{(m)} R_{ijkl}$ by an expression $(+-)\nabla^{(m+2)}\phi_{u+1} g_{ab}$ and then replacing the two indices $({}_c, {}_d)$ in the second factor (which now contract against each other, by virtue of the term g_{ab} that we brought out) by an expression $(\nabla^d \omega, {}_d)$.[111] We also denote by $Rep^2[C_g]$ the complete contraction that formally arises from C^v_g by replacing the first factor $\nabla^{(m)} R_{ijkl}$ by an expression $(+-)\nabla^{(m+2)}\omega g_{ab}$,[112] and then replacing the two indices $({}_c, {}_d)$ in the second factor by an expression $(\nabla^t \phi_{u+1}, {}_t)$. We then define (slightly abusing notation):

$$Rep^1[C^v_g] = \sum_{[(a,b),(c,d)] \in INT^2} Rep^1_{[(a,b),(c,d)]}[C^v_g], \tag{7.29}$$

and we similarly define $Rep^2[C^v_g]$.

It follows by definition that:

$$Sub^{\sigma+u+1}_{\omega}\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\} = \sum_{v \in V} a_v (Rep^1 + Rep^2)[C^v_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)]. \tag{7.30}$$

Thus, we derive that:

$$\begin{aligned} &Sub^{\sigma+u+1}_{\phi_{u+1}}\{Image^{1,\beta}_{\phi_{u+1}}[L_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)]\} = \\ &\sum_{l \in L} a_l (Rep^1 + Rep^2)[Xdiv_{i_1} \dots Xdiv_{i_a} C^{l,i_1 \dots i_a}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)] \\ &+ \sum_{j \in J} a_j (Rep^1 + Rep^2)[C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)]. \end{aligned} \tag{7.31}$$

Now, we will be separately studying each of the sublinear combinations

$$Sub^{\sigma+u+1}_{\omega}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1} \dots Xdiv_{i_a} C^{l,i_1 \dots i_a}_g]\},$$

$Sub^{\sigma+u+1}_{\omega}\{Image^{1,\beta}_{\phi_{u+1}}[C^j_g]\}$. By the definition of the formal operation Rep we derive:

$$\sum_{j \in J} a_j Sub^{\sigma+u+1}_{\omega}\{Image^{1,\beta}_{\phi_{u+1}}[C^j_g]\} = \sum_{j \in J^{\sigma+u+1}} a_j C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u, \phi_{u+1}, \omega). \tag{7.32}$$

[111] Recall that ${}_c$ is a derivative index, so this formal operation is well-defined–in other words we may *formally* erase the index ${}_c$ and bring in a new factor $\nabla\omega$ which will then contract against the index ${}_d$.

[112] See (2.1).

On the other hand, in order to understand each sublinear combination $Sub^{\sigma+u+1}_{\omega}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g]\}$, we firstly seek to write out that linear combination in a *normalized form*: We impose the restriction that all the factors $\nabla\phi_1,\dots,\nabla\phi_{u+1}$ or $\nabla\omega$ that are contracting against a factor $\nabla^{(A)}\Omega_f$ $(A\ge 2)$ in the tensor fields appearing in the RHSs of all equations until (7.37) must be contracting against the leftmost indices. If this condition does not hold for some contraction that appears in $Sub^{\sigma+u+1}_{\phi_{u+1}}\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}$, we apply the curvature identity enough times to make it hold. (We will refer to this below as the *shifting operation*).

We then distinguish three cases regarding the two internal indices $({}_a,{}_b)$ discussed in the definition above: Either in our tensor field $C^{l,i_1\dots i_a}_g$ neither of the indices ${}_a$, ${}_b$ is a free index, or precisely one of them is a free index, or that both of them are free indices. We will accordingly denote those sublinear combinations by

$$Sub^{\sigma+u+1,I}_{\omega}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g]\},$$

$$Sub^{\sigma+u+1,II}_{\omega}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g]\},$$

$$Sub^{\sigma+u+1,III}_{\omega}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g]\}.$$

Then, for each $l\in L$, by the above discussion, we calculate:

$$\begin{gathered}Sub^{\sigma+u+1,I}_{\omega}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g]\}=\\ \sum_{u\in U_1\bigcup U_1^\sharp}a_u Xdiv_{i_1}\dots Xdiv_{i_a}C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+\\ \sum_{u\in U_2\bigcup U_2^\sharp}a_u Xdiv_{i_1}\dots Xdiv_{i_a}C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+\\ \sum_{m\in M\bigcup M^\sharp}a_m C^{m,i_1\dots i_{a+2}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{a+1}}\phi_{u+1}\nabla_{i_{a+2}}\omega+\\ \sum_{j\in J^{\sigma+u+1}}a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1},\omega).\end{gathered}\tag{7.33}$$

Now, we seek to study each $Sub^{\sigma+u+1,II}_{\omega}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g]\}$. We will draw a different conclusion, in the cases where $l\in L_\mu$ and where $l\in L\setminus L_\mu$. In general, for any $l\in L$, we make special note of the factors $S_*\nabla^{(\nu)}R_{ijkl}$ in $C^{l,i_1\dots i_a}_g$ that contain free indices. By the hypothesis of Lemma 1.3 $l\in L_\mu$, those free indices will be of the form ${}_{r_1},\dots,{}_{r_\nu},{}_j$.

Definition 7.3 *We denote the set of free indices that belong to factors $S_*\nabla^{(\nu)}R_{ijkl}$ with $\nu>0$ by $I^\sharp$. For future reference we denote by $I^\sharp_*$ the set of free indices that belong to a factor $S_*\nabla^{(\nu)}R_{ijkl}$ with $\nu=0$.*

Recall that for tensor fields $C^{l,i_1\dots i_\mu}_g, l\in L_\mu$ we will have $I^\sharp_* = \emptyset$, by virtue of the assumptions in the beginning of this paper.

Now, another definition:

Definition 7.4 *For each ${}_{i_h}\in I^\sharp$, we denote by $C^{l,i_1\dots i_a|f(i_h)}_g$ the tensor field that arises from $C^{l,i_1\dots i_a}_g$ by replacing the factor $T({}_{i_h}) = S_*\nabla^{(\nu)}_{i_h r_2\dots r_\nu}R_{ijkl}$ to which ${}_{i_h}$ belongs (assume with no loss of generality that ${}_{i_h} = {}_{r_1}$) by a factor $\frac{1}{\nu+1}\nabla^{(\nu)}_{r_2\dots r_{\nu-1}j}R_{ii_hkl}$.*

Then, for each ${}_{i_h}\in I^\sharp\bigcup I^\sharp_*$, we denote by

$$Rep^{i_h,1,\phi_{u+1},\omega}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

the $(a-1)$-tensor field that arises from $C^{l,i_1\dots i_a|f(i_h)}_g$ by replacing the factor $\frac{1}{\nu+1}\nabla^{(\nu)}_{r_2\dots r_{\nu-1}j}R_{ii_hkl}$ by a factor $\frac{1}{\nu+1}\nabla^{(\nu+2)}_{r_2\dots r_{\nu-1}jil}\omega$ and then making the index ^{k} contract against a factor $\nabla_k\phi_{u+1}$. We also denote by

$$Rep^{i_h,1,\omega,\phi_{u+1}}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

the $(a-1)$-tensor field that arises from

$$Rep^{i_h,1,\phi_{u+1},\omega}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

by replacing ϕ_{u+1} by ω and ω by ϕ_{u+1}.

Analogously, we denote by $Rep^{i_h,2,\phi_{u+1},\omega}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ the $(a-1)$-tensor field that arises from $C^{l,i_1\dots i_a|f(i_h)}_g$ by replacing the factor $\frac{1}{\nu+1}\nabla^{(\nu)}_{r_2\dots r_{\nu-1}j}R_{ii_hkl}$ by a factor $-\frac{1}{\nu+1}\nabla^{(\nu+2)}_{r_2\dots r_{\nu-1}jik}\omega$ and then making the index ^{l} contract against a factor $\nabla_l\phi_{u+1}$. We also denote by $Rep^{i_h,2,\omega,\phi_{u+1}}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ the $(a-1)$-tensor field that arises from

$$Rep^{i_h,2,\phi_{u+1},\omega}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

by replacing ϕ_{u+1} by ω and ω by ϕ_{u+1}.

Finally, for each ${}_{i_h}\in I^\sharp$ we denote by

$$FRep^{i_h,1,\phi_{u+1},\omega}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

the $(a-1)$ tensor fields that arises from

$$Rep^{i_h,1,\phi_{u+1},\omega}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$

by replacing the expression $\frac{1}{\nu+1}\nabla^{(\nu+2)}_{r_2\dots r_{\nu-1}jik}\omega\nabla^i\phi_f,\dots$ by an expression $\frac{1}{\nu+1}S_*\nabla^{(\nu-1)}_{r_2\dots r_{\nu-1}}R_{ijlk}\nabla^l\omega\nabla^i\phi_f$. (This is well-defined since ${}_{i_h}\in I^\sharp$, therefore $\nu>0$). Accordingly we define $FRep^{i_h,2,\phi_{u+1},\omega}$, $FRep^{i_h,1,\omega,\phi_{u+1}}$, $FRep^{i_h,2,\omega,\phi_{u+1}}$.

Then, in view of the hypotheses of Lemma 1.3, and by applying the shifting operation to $\nabla^{(\nu+2)}\omega$, we derive that for each $l \in L_\mu$:

$$
\begin{aligned}
&Sub_\omega^{\sigma+u+1,II}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_\mu} C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}\\
&=\sum_{i_h\in I^\sharp} Xdiv_{i_1}\dots X\hat{div}_{i_h}\dots Xdiv_{i_\mu}\\
&\{FRep^{i_h,1,\omega,\phi_{u+1}}[C^{l,i_1\dots i_\mu|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\\
&+FRep^{i_h,1,\phi_{u+1},\omega}[C^{l,i_1\dots i_\mu|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\\
&+FRep^{i_h,2,\omega,\phi_{u+1}}[C^{l,i_1\dots i_\mu|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&FRep^{i_h,2,\phi_{u+1},\omega}[C^{l,i_1\dots i_\mu|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}+\\
&\sum_{j\in J^{\sigma+u+1}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega).
\end{aligned}
\tag{7.34}
$$

On the other hand, by virtue of our assumptions on the tensor fields in (1.6) and by the analysis above, we easily see that for each $l \in L \setminus L_\mu$:

$$
\begin{aligned}
&Sub_\omega^{\sigma+u+1,II}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}=\\
&\sum_{u\in U_1} a_u Xdiv_{i_1}\dots Xdiv_{i_a} C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+\\
&\sum_{u\in U_2} a_u Xdiv_{i_1}\dots Xdiv_{i_a} C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+\\
&\sum_{m\in M} a_m C^{m,i_1\dots i_{a+2}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{a+1}}\phi_{u+1}\nabla_{i_{a+2}}\omega+\\
&\sum_{j\in J^{\sigma+u+1}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega).
\end{aligned}
\tag{7.35}
$$

Finally, we study the linear combinations

$$Sub_\omega^{\sigma+u+1,III}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g]\}.$$

Clearly, by the hypothesis of Lemma 1.3 that no μ-tensor fields have special free indices, we derive that for each $l \in L_\mu$:

$$Sub_\omega^{\sigma+u+1,III}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g]\}=0.$$

In addition, for each $l \in L_k, k \geq \mu+2$ we straightforwardly obatin:

$$
\begin{aligned}
&Sub_\omega^{\sigma+u+1,III}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}\\
&=\sum_{u\in U_1} a_u Xdiv_{i_1}\dots Xdiv_{i_a}C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+\\
&\sum_{u\in U_2} a_u Xdiv_{i_1}\dots Xdiv_{i_a}C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+\\
&\sum_{m\in M} a_m C^{m,i_1\dots i_{a+2}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{a+1}}\phi_{u+1}\nabla_{i_{a+2}}\omega+\\
&\sum_{j\in J^{\sigma+u+1}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega).
\end{aligned}
\tag{7.36}
$$

So, we next set out to study the linear combination

$$Sub_\omega^{\sigma+u+1,III}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}$$

for each $l\in L_{\mu+1}$. We just introduce one more piece of notation:

Definition 7.5 *We denote by* $\sum_{b\in B} a_b C^{b,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)$ *a generic linear combination of acceptable* $(\mu-1)$*-tensor fields in the form (1.5), with length* $\sigma+u$*, weight* $-n+\mu+3$*, and a u-simple character that arises from* $\vec{\kappa}_{simp}$ *by formally replacing a factor* $\nabla^{(m)}R_{ijkl}$ *by* $\nabla^{(m+2)}Y$*. We denote the other factors other than* $\nabla\phi$*'s in* $C^{z,i_1\dots i_{\mu-1}}_g$ *by* $F_1,\dots,F_{\sigma-1}$*.*

We then define $Hit^K_\omega[C^{b,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)]$ *to stand for the* $(\mu-1)$*-tensor field that arises from* $C^{b,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)$ *by hitting the factor* F^K *by a derivative* ∇_{i_*} *and then contracting* $_{i_*}$ *against a factor* $\nabla\omega$*.*

We denote by

$$\sum_{\zeta\in Z} a_\zeta C^{\zeta,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})$$

a generic linear combination of $(\mu-1)$*-acceptable tensor fields with weight* $-n+\mu+3$*, length* $\sigma+u+1$ *and a u-simple character* $\vec{\kappa}_{simp}$ *and with one factor* $\nabla\phi_{u+1}$ *contracting against a factor* $\nabla^{(\nu)}R_{ijkl}$*. We denote this factor* $\nabla^{(\nu)}R_{ijkl}$ *by* F *and we denote the other factors other than* $\nabla\phi$*'s in* $C^{\zeta,i_1\dots i_{\mu-1}}_g$ *by* $F_1,\dots,F_{\sigma-1}$*.*

We then define $Hit^K_\omega[C^{\zeta,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})]$ *to stand for the* $(\mu-1)$*-tensor field that arises from* $C^{\zeta,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})$ *by hitting the factor* F^K *by a derivative* ∇_{i_*} *and then contracting* $_{i_*}$ *against a factor* $\nabla\omega$ *(or* $\nabla\phi_{u+1}$*).*

Two types of linear combinations that we will be encountering are linear combinations in the forms:

$$\sum_{b\in B} a_b \sum_{K=1}^{\sigma-1} Hit^K_\omega[C^{b,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)],$$

$$\sum_{\zeta\in Z} a_\zeta \sum_{K=1}^{\sigma-1} Hit^K_\omega[C^{\zeta,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})].$$

For complete contractions as above, we denote by

$$Switch\{Hit^K_\omega[C^{b,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)]\}$$

the tensor fields that arise by interchanging the functions ϕ_{u+1},ω.

Now, by the same argument as for equation (7.34), we derive that for each $l\in L_{\mu+1}$:

$$\begin{aligned}
&Sub^{\sigma+u+1,III}_\omega\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\} = \\
&\sum_{b\in B} a_b Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} \sum_{K=1}^{\sigma-1}\{Hit^K_\omega[C^{b,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)] \\
&Switch\{Hit^K_\omega[C^{b,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)]\}\}+ \\
&\sum_{m\in M} a_m Xdiv_{i_1}\dots Xdiv_{i_a} C^{m,i_1\dots i_{a+2}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{a+1}}\phi_{u+1}\nabla_{i_{a+2}}\omega \\
&+\sum_{j\in J^{\sigma+u+1}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega).
\end{aligned} \tag{7.37}$$

In conclusion, combining equations (7.7), (7.32), (7.33), (7.34), (7.35), (7.36), (7.37), and replacing them into (7.27), we have shown that:

$$
\begin{aligned}
&\sum_{l\in L} a_l Sub_\omega^{\sigma+u+2}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}\\
&+\sum_{j\in J} a_j Sub_\omega^{\sigma+u+2}\{Image^{1,\beta}_{\phi_{u+1}}[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}+\\
&\sum_{l\in L_\mu} a_l \sum_{i_h\in I^\sharp} Xdiv_{i_1}\dots \hat{Xdiv}_{i_h}\dots Xdiv_{i_\mu}\\
&\{FRep^{i_h,1,\omega,\phi_{u+1}}[C^{l,i_1\dots i_\mu|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\\
&+\sum_{l\in L_\mu} a_l FRep^{i_h,1,\phi_{u+1},\omega}[C^{l,i_1\dots i_\mu|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\\
&+\sum_{l\in L_\mu} a_l FRep^{i_h,2,\omega,\phi_{u+1}}[C^{l,i_1\dots i_\mu|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&\sum_{l\in L_\mu} a_l FRep^{i_h,2,\phi_{u+1},\omega}[C^{l,i_1\dots i_\mu|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}+\\
&\sum_{b\in B} a_b Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} \sum_{K=1}^{\sigma-1}\{Hit^K_\omega[C^{b,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)]+\\
&Switch\{Hit^K_\omega[C^{b,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)]\}\}+\\
&\sum_{u\in U_1\bigcup U_1^\sharp} a_u Xdiv_{i_1}\dots Xdiv_{i_a} C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+\\
&\sum_{u\in U_2\bigcup U_2^\sharp} a_u Xdiv_{i_1}\dots Xdiv_{i_a} C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+\\
&\sum_{m\in M} a_m Xdiv_{i_1}\dots Xdiv_{i_a} C^{m,i_1\dots i_{a+2}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{a+1}}\phi_{u+1}\nabla_{i_{a+2}}\omega\\
&+\sum_{j\in J^{\sigma+u+1}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+\\
&\sum_{\zeta\in Z} a_\zeta Xdiv_{i_1}\dots Xdiv_{\mu-1} Hit^K_\omega[C^{\zeta,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})]\\
&=\sum_{t\in T^{\sigma+u+2}} a_t C^t_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega),
\end{aligned}
\tag{7.38}
$$

modulo complete contractions of length $\geq \sigma+u+3$. The sublinear combination in the RHS is a generic sublinear combination as defined below (7.2). Notice that the minimum length among the complete contractions above is $\sigma+u+1$. The complete contractions (and tensor fields) with $\sigma+u+1$ factors are indexed in $U_1, U_1^\sharp, U_2, U_2^\sharp, B, J^{\sigma+u+1}$.

Therefore the above equation implies:

$$
\sum_{b\in B} a_b Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} \sum_{K=1}^{\sigma-1}\{Hit^K_\omega[C^{b,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)]
$$

$$
+ Switch\{Hit^K_\omega[C^{b,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)]\}\}+
$$

$$
\sum_{u\in U_1\bigcup U_1^\sharp} a_u Xdiv_{i_1}\dots Xdiv_{i_a} C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+
$$

$$
\sum_{u\in U_2\bigcup U_2^\sharp} a_u Xdiv_{i_1}\dots Xdiv_{i_a} C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+
$$

$$
\sum_{j\in J^{\sigma+u+1}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)=0, \tag{7.39}
$$

modulo complete contractions of length $\ge \sigma+u+2$. We claim that in the *non-special cases*:

$$
\sum_{b\in B} a_b Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} \sum_{K=1}^{\sigma-1}\{Hit^K_\omega[C^{b,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)]
$$

$$
+ Switch\{Hit^K_\omega[C^{b,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)]\}\} =
$$

$$
\sum_{u\in U_1\bigcup U_1^\sharp} a_u Xdiv_{i_1}\dots Xdiv_{i_a} C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+
$$

$$
\sum_{u\in U_2\bigcup U_2^\sharp} a_u Xdiv_{i_1}\dots Xdiv_{i_a} C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+
$$

$$
\sum_{j\in J^{\sigma+u+1}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+
$$

$$
\sum_{\zeta\in Z} a_\zeta Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}}\{\sum_{K=1}^{\sigma-1} Hit^K_\omega[C^{\zeta,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})]
$$

$$
+ Switch[\sum_{K=1}^{\sigma-1} Hit^K_\omega[C^{\zeta,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})]]\}+
$$

$$
\sum_{m\in M} a_m C^{m,i_1\dots i_{a+2}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{a+1}}\phi_{u+1}\nabla_{i_{a+2}}\omega+
$$

$$
\sum_{j\in J^{\sigma+u+2}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)
$$

$$
=\sum_{t\in T} a_t C^t_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega), \tag{7.40}
$$

modulo complete contractions of length $\ge \sigma+u+3$. Here the terms indexed

in U_1, U_2 in the RHS are *generic* linear combination in the forms described in Definition 7.2. In the special cases, we claim:

$$\sum_{b\in B} a_b Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}}\sum_{K=1}^{\sigma-1}\{Hit^K_\omega[C^{b,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)]$$
$$+ Switch\{Hit^K_\omega[C^{b,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)]\}\}+$$
$$\sum_{u\in U_1\bigcup U_1^\sharp} a_u Xdiv_{i_1}\dots Xdiv_{i_a} C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+$$
$$\sum_{u\in U_2\bigcup U_2^\sharp} a_u Xdiv_{i_1}\dots Xdiv_{i_a} C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+$$
$$\sum_{j\in J^{\sigma+u+1}\bigcup J^{\sigma+u+2}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)$$
$$= \sum_{m\in M_{\mu-1}} a_m C^{m,i_1\dots i_{mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega). \tag{7.41}$$

Here the tensor fields indexed in $M_{\mu-1}$ are acceptable, have length $\sigma+u+2$, u-simple character $\vec{\kappa}_{simp}$ and moreover each of the $\mu-1$ free indices belongs to a different factor.

The harder challenge is to prove (7.40), so we start with that equation.

Proof of (7.40): Let us pick out the sublinear combination in (7.39) with a factor $\nabla\omega$ contracting against a given factor F_1.[113] Since this sublinear combination must vanish separately, we derive that:

$$\sum_{b\in B} a_b Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} Hit^1_\omega[C^{b,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)]+$$
$$\sum_{u\in \overline{U}_1\bigcup \overline{U}_1^\sharp} a_u Xdiv_{i_1}\dots Xdiv_{i_a} C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+$$
$$\sum_{j\in J^{\sigma+u+1}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)=0, \tag{7.42}$$

modulo complete contractions of length $\geq \sigma+u+2$. Here the index sets $\overline{U}_1\subset U_1, \overline{U}_1^\sharp\subset U_1^\sharp$, are the index sets of terms with a factor $\nabla\omega$ contracting against the factor F_1.

Our aim is to derive an equation like the above, only with the factor $\nabla\omega$ contracting against a derivative index in the factor F_1, and moreover, if F_1 is

[113] We assume for convenience that F_1 is a well-defined factor in $\vec{\kappa}_{simp}$. If it were not, we just pick out the sublinear combination where $\nabla\phi_{u+1}$ contracts against any generic factor $\nabla^{(m)}R_{ijkl}$ and the same argument applies.

of the form $\nabla^{(B)}\Omega_h$, then we additionaly require that $B \geq 3$. Call this the $*$-property. Now, if F_1 is a curvature factor, we apply the inductive assumption of Lemma 4.10 in [6] to ensure that in all terms in $\overline{U}_1$ the factor $\nabla\phi_{u+1}$ is not contracting against a special index. Now, if F_1 is a factor $\nabla^{(B)}\Omega_x$, we apply the inductive assumption of Lemma 4.1 in [6] if necessary to assume wlog that $\overline{U}_1^\sharp = \emptyset$. Finally, if needed, we apply the inductive assumptions onf Corollaries 2 or 3 in [6] (if F_1 is a simple factor in the form $S_*\nabla^{(\nu)}R_{ijkl}$, or a simple factor in the form $\nabla^{(A)}\Omega_h$, respectively) to ensure that for each $u \in \overline{U}_1$ $\nabla\omega$ is not contracting against a factor S_*R_{ijkl} or $\nabla^{(2)}\Omega_h$.[114] Therefore, we may assume wlog that the $*$-property holds in (7.42).

We then apply the Eraser to the factor $\nabla\phi_{u+1}$ (see the Appendix in [3]) and derive a new equation:

$$\begin{aligned}
&\sum_{b\in B} a_b Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} C^{b,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)+\\
&\sum_{u\in\overline{U}_1\bigcup\overline{U}_1^\sharp} a_u Xdiv_{i_1}\dots Xdiv_{i_a} Erase_\omega[C^{u,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)]+\\
&\sum_{j\in J^{\sigma+u+1}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})=0,
\end{aligned}\tag{7.43}$$

modulo complete contractions of length $\geq \sigma+u+1$.

Now, apply the inductive assumption of Corollary 1 in [6] to the above. We derive that there exists a linear combination of acceptable μ-tensor fields with a simple character $Pre(\vec{\kappa}_{simp})$ (indexed in P below) such that:

$$\begin{aligned}
&\sum_{b\in B} a_b C^{b,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon+\\
&\sum_{p\in P} a_p Xdiv_{i_\mu} C^{p,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon=\\
&\sum_{j\in J^{\sigma+u+1}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\upsilon^{\mu-1})=0,
\end{aligned}\tag{7.44}$$

modulo complete contractions of length $\geq \sigma+u+\mu$ (the terms in the above have lengh $\sigma+u+\mu-1$). Then, keeping track of the greater length correction terms that arise in the above, we derive a new equation:

[114] In all the above applications of Lemmas and Corollaries from [6], we observe that by weight considerations, the fact that (1.6) does not fall under the special cases ensures that there is no danger of falling under a forbidden case of those Lemmas/Corollaries.

$$\sum_{b \in B} a_b C_g^{b,i_1 \dots i_{\mu-1}}(\Omega_1, \dots, \Omega_p, \phi_{u+1}, \phi_1, \dots, \phi_u) \nabla_{i_1} \upsilon \dots \nabla_{i_{\mu-1}} \upsilon +$$
$$\sum_{p \in P} a_p X div_{i_\mu} C_g^{p,i_1 \dots i_\mu}(\Omega_1, \dots, \Omega_p, \phi_{u+1}, \phi_1, \dots, \phi_u) \nabla_{i_1} \upsilon \dots \nabla_{i_{\mu-1}} \upsilon +$$
$$\sum_{j \in J^{\sigma+u+1}} a_j C_g^j(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u, \phi_{u+1}, \upsilon^{\mu-1}) = \tag{7.45}$$
$$\sum_{\zeta \in Z} a_\zeta C_g^{\zeta,i_1 \dots i_{\mu-1}}(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u, \phi_{u+1}) \nabla_{i_1} \upsilon \dots \nabla_{i_{\mu-1}} \upsilon +$$
$$\sum_{j \in J} a_j C_g^{j,i_1 \dots i_{\mu-1}}(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u, \phi_{u+1}) \nabla_{i_1} \upsilon \dots \nabla_{i_{\mu-1}} \upsilon.$$

This equation holds *perfectly*–not modulo longer terms. The terms indexed in J have length $\sigma + u + \mu$, a factor $\nabla\omega$ and a u-simple character $\vec{\kappa}_{simp}$; the terms indexed in T, a factor $\nabla^{(B)}\phi_{u+1}$. Thus, invoking the last Lemma in the Appendix of [3], we derive:

$$\sum_{b \in B} a_b X div_{i_1} \dots X div_{i_{\mu-1}} C_g^{b,i_1 \dots i_{\mu-1}}(\Omega_1, \dots, \Omega_p, \phi_{u+1}, \phi_1, \dots, \phi_u) +$$
$$\sum_{p \in P} a_p X div_{i_1} \dots X div_{i_{\mu-1}} X div_{i_\mu} C_g^{p,i_1 \dots i_\mu}(\Omega_1, \dots, \Omega_p, \phi_{u+1}, \phi_1, \dots, \phi_u) +$$
$$\sum_{j \in J^{\sigma+u+1}} a_j C_g^j(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u, \phi_{u+1}) =$$
$$\sum_{\zeta \in Z} a_\zeta X div_{i_1} \dots X div_{i_{\mu-1}} C_g^{\zeta,i_1 \dots i_{\mu-1}}(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u, \phi_{u+1}) +$$
$$\sum_{j \in J} a_j C_g^{j,i_1 \dots i_{\mu-1}}(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u, \phi_{u+1}). \tag{7.46}$$

Thus, by operating on the above by the operation $\sum_{K=1}^{\sigma-1} Hit^K[\dots]$ (this clearly produces a true equation), and then interchanging the two functions ϕ_{u+1}, ω (this also produces a new true equation), we derive (7.40).

Proof of (7.41): We just neglect the algebraic structure of $\sum_{K=1}^{\sigma-1} Hit^K_\omega$ in the LHS and apply the Lemma 4.10 in [6] to (7.39. We use the fact that the LHS of the resulting equation vanishes formally at the linearized level, and then repeat the formal permutations of indices to the non-linearized level, and finaly replace the $\mu-1$ factors $\nabla\upsilon$ by $X div$'s (using the last Lemma in the Appendix of [3]). □

Therefore, replacing (7.40) (or (7.41)) into (7.38) we derive:

$$
\begin{aligned}
&\sum_{l\in L} a_l Sub_\omega^{\sigma+u+2}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}+\\
&Sub_\omega^{\sigma+u+2}\{Image^{1,\beta}_{\phi_{u+1}}[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}\\
&+\sum_{l\in L_\mu} a_l \sum_{i_h\in I^\sharp} Xdiv_{i_1}\dots X\hat{div}_{i_h}\dots Xdiv_{i_\mu}\\
&\{FRep^{i_h,1,\omega,\phi_{u+1}}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\\
&+\{FRep^{i_h,1,\phi_{u+1},\omega}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\\
&+FRep^{i_h,2,\omega,\phi_{u+1}}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&FRep^{i_h,2,\phi_{u+1},\omega}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}\}+\\
&(\sum_{m\in M_{\mu-1}} a_m Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}}C^{m,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_\mu}\phi_{u+1}\nabla_{i_{\mu+1}}\omega)\\
&+\sum_{m\in M} a_m Xdiv_{i_1}\dots Xdiv_{i_a}C^{m,i_1\dots i_{a+2}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{a+1}}\phi_{u+1}\nabla_{i_{a+2}}\omega+\\
&\sum_{\zeta\in Z} a_\zeta Xdiv_{i_1}\dots Xdiv_{\mu-1}Hit^K_\omega[C^{\zeta,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})]+\\
&\sum_{j\in J^{\sigma+u+2}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)\\
&=\sum_{t\in T} a_t C^t_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega),
\end{aligned}
\tag{7.47}
$$

modulo complete contractions of length $\geq \sigma+u+3$.

Hence, we are reduced to studying the sublinear combinations

$$
Sub_\omega^{\sigma+u+2}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}
\tag{7.48}
$$

and

$$
Sub_\omega^{\sigma+u+2}\{Image^{1,\beta}_{\phi_{u+1}}[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}.
$$

As before, we straightforwardly derive:

$$
\begin{aligned}
&Sub_\omega^{\sigma+u+2}\{Image^{1,\beta}_{\phi_{u+1}}[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}\\
&=\sum_{j\in J^{\sigma+u+2}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega).
\end{aligned}
\tag{7.49}
$$

To analyze the sublinear combination (7.48) we firstly seek to understand how it arizes:

A study of the sublinear combination $Sub^{\sigma+u+2}_\omega\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}$**:**

As before, we write out $Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ as a linear combination of complete contractions, say

$$Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots,i_a}_g=\sum_{x\in X}a_xC^x_g. \tag{7.50}$$

Then, for each C^x_g we identify the *(ordered) sets of pairs of pairs of indices*, $[({}_a,{}_b),({}_c,{}_d)]$ where ${}_a,{}_b$ belong to the same factor and ${}_c,{}_d$ belong to the same factor, and either ${}_a$ is contracting against ${}_c$ and ${}_b$ against ${}_d$ on vice versa, *and both* ${}_a,{}_c$ *are derivative indices.* Denote this set of ordered pairs by Z^x. Then, for each $[({}_a,{}_b),({}_c,{}_d)]\in Z^x$ we let $B_{[(a,b),(c,d)]}\{C^x_g\}$ stand for the complete contraction that formally arises from C^x_g by applying the last summand in (2.2) to the indices $({}_a,{}_b)$ (recall that one of them is a derivative index, so this is a well-defined operation), thus making the indices ${}_c,{}_d$ contract against each other. Then apply Sub_ω to this complete contraction we have created. This replaces the internal contraction between ${}_c,{}_d$ by a factor $\nabla\omega$ (since ${}_c$ is a derivative index). Denote the complete contraction we thus obtain by $\overline{B}_{[(a,b),(c,d)]}\{C^x_g\}$. It follows by the definition of $Sub^{\sigma+u+2}_\omega\{\dots\}$ that:

$$\begin{aligned}&Sub^{\sigma+u+2}_\omega\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}=\\&\sum_{x\in X}a_x\sum_{[(a,b),(c,d)]\in Z^x}\overline{B}_{[(a,b),(c,d)]}\{C^x_g\}.\end{aligned} \tag{7.51}$$

Having obtained an understanding of how

$$Sub^{\sigma+u+2}_\omega\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}$$

arises, we now proceed to express it in a more useful form:

We distinguish cases depending on the *form* of the indices $({}_a,{}_b),({}_c,{}_d)$, for each complete contraction $C_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ appearing in

$$Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u).$$

Recall that we have defined an index in C_g to be a *divergence index* if it is an index ∇^{i_f} which has arisen by taking an $Xdiv$ operation, $Xdiv_{i_f}$, with respect to some free index ${}_{i_f}$; we have also defined an index in C_g to be an *original index* in $C^{l,i_1\dots,i_a}_g$ if the index appears in the tensor field $C^{l,i_1\dots,i_a}_g$ (before we take any $Xdiv$'s).

Now, we place each complete contraction in

$$Sub^{\sigma+u+2}_\omega\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}$$

into one of the sublinear combinations

$$Sub_\omega^{\sigma+u+2,K}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots,i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}$$

$(K=\alpha,\beta,\gamma,\delta)$ based on the pair $(a,b),(c,d)$ from which it arose.

Specifically:

Definition 7.6 *Refer to (7.51) and pick out a term in the RHS. For any given index* ${}_a,{}_b,{}_c,{}_d$ *(recall that we are now assuming that* ${}_a,{}_c$ *are derivative indices), we inquire whether it is an original index or a divergence index* $\nabla^{i_h}, h=1,\dots,a$*. Accordingly, we place the term* $\overline{B}_{[(a,b),(c,d)]}\{C^x_g\}$ *into one of the four sublinear combinations* $Sub_\omega^{\sigma+u+2,\alpha}$, $Sub_\omega^{\sigma+u+2,\beta}$, $Sub_\omega^{\sigma+u+2,\gamma}$, $Sub_\omega^{\sigma+u+2,\delta}$ *according to the following rule:*

We declare that C_g *belongs to* $Sub_\omega^{\sigma+u+2,\alpha}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots,i_a}_g]\}$ *if and only if* ${}_a$ *and* ${}_c$ *are divergence indices.*

We declare that C_g *belongs to* $Sub_\omega^{\sigma+u+2,\beta}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots,i_a}_g]\}$ *if and only if only one of the indices* ${}_a,{}_b,{}_c,{}_d$ *is a divergence index (say* ${}_a$ *with no loss of generality). We declare that* C_g *belongs to* $Sub_\omega^{\sigma+u+2,\gamma}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots,i_a}_g]\}$ *if either* ${}_a,{}_b$ *or* ${}_c,{}_d$ *are both divergence indices. Finally, we declare that* C_g *belongs to* $Sub_\omega^{\sigma+u+2,\delta}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots,i_a}_g]\}$ *if all four indices* ${}_a,{}_b,{}_c,{}_d$ *are original indices.*

Now, another piece of notation: We denote by

$$\sum_{m\in M^{\sharp\sharp}} a_m C^{m,i_1\dots i_{a+2}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{a+1}}\phi_{u+1}\nabla_{i_{a+2}}\omega$$

a generic linear combination of tensor fields of length $\sigma+u+2$ with *two* unnormalized factors $\nabla\Omega_h,\nabla\Omega_{h'}$, that are contracting against factors $\nabla\phi_{u+1}$, $\nabla\omega$ respectively. We also require that if $a=\mu$ then all free indices must be nonspecial. Now, the first thing we easily notice is that for each $l\in L$:

$$Sub_\omega^{\sigma+u+2,\delta}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots,i_a}_g]\}=\sum_{m\in M\bigcup M^\sharp\bigcup M^{\sharp\sharp}} Xdiv_{i_1}\dots Xdiv_{i_a}a_m C^{m,i_1\dots i_{a+2}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{a+1}}\phi_{u+1}\nabla_{i_{a+2}}\omega. \tag{7.52}$$

In order to describe $Sub_\omega^{\sigma+u+2,\alpha}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots,i_a}_g]\}$, we define $(I\times I)^\sharp$ to stand for the subset of $(I\times I)$ that consists of all ordered pairs of free indices that belong to different factors. For each $l\in L$ we then compute:

$$Sub_\omega^{\sigma+u+2,\alpha}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots,i_a}_g]\}=\sum_{(i_k,i_l)\in(I\times I)^\sharp} Xdiv_{i_1}\dots X\hat{d}iv_{i_k}\dots X\hat{d}iv_{i_l}\dots Xdiv_{i_a}C^{l,i_1\dots,i_a}_g\nabla_{i_k}\phi_{u+1}\nabla_{i_l}\omega. \tag{7.53}$$

In particular, we observe that if $l \in L_K, K \geq \mu + 2$, the right hand side of the above is a generic linear combination of the form:

$$\sum_{u \in U_1} a_u X div_{i_1} \ldots X div_{i_a} C_g^{u,i_1 \ldots, i_a}(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u, \phi_{u+1}, \omega) + \\ \sum_{u \in U_2} a_u X div_{i_1} \ldots X div_{i_a} C_g^{u,i_1 \ldots, i_a}(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u, \phi_{u+1}, \omega). \tag{7.54}$$

Now, to describe each $Sub_\omega^{\sigma+u+2,\beta}\{Image^{1,\beta}_{\phi_{u+1}}[X div_{i_1} \ldots X div_{i_a} C_g^{l,i_1 \ldots, i_a}]\}$, we introduce more notation: For each $l \in L_\mu$ and each ${}_{i_h} \in I_l$,[115] we define $T(i_h)$ to stand for the factor to which ${}_{i_h}$ belongs. We observe that if $l \in L_\mu$ then for each factor $T(i_h)$ of the form $T(i_h) = \nabla^{(m)} R_{ijkl}$ or $T(i_h) = \nabla^{(A)}\Omega_f$, ${}_{i_h}$ must be a derivative index, since in the setting of Lemma 1.3 no μ-tensor field contains special free indices. If $T(i_h)$ is of the form $S_*\nabla^{(\nu)} R_{ijkl}$, we then have that ${}_{i_h}$ must be one of the indices ${}_{r_1}, \ldots, {}_{r_\nu}, {}_j$ (by the first assumption in the introduction). In that case, we write out $T(i_h)$ as a sum of tensors of the form $\nabla^{(\nu)} R_{ijkl}$.

Definition 7.7 *With the above convention, for each $l \in L$,[116] we denote by $I^*_l \subset I_l$[117] the set of free indices that are derivative indices. We denote by $I^+_l = I_l \setminus I^*_l$.[118] For each ${}_i \in I^*_l$, we denote by $Set(T(i))$[119] to be the set of all the indices in $T(i)$ that are not free and not contracting against a factor $\nabla\phi_h$. For each ${}_i \in I^+_l$, we denote by $Set(T(i))$ the set of derivative indices in the factor $\nabla^{(\nu)} R_{ijkl}$ that are not free and not contracting against a factor $\nabla\phi_h$.*

Then, for each ${}_i \in I^*_l$ and each $t \in Set(T(i))$, let $Repla^{i,t}_{\phi_{u+1},\omega}[C_g^{l,i_1 \ldots, i_a}]$ be the $(a-1)$ tensor field that formally arises from $C_g^{l,i_1 \ldots, i_a}$ by erasing the index ${}_i$ and making the index ${}_t$ contract against a factor $\nabla\phi_{u+1}$ and also making the index ^{t} contract against a factor $\nabla\omega$. We denote by $Repla^{i,t}_{\omega,\phi_{u+1}}[C_g^{l,i_1 \ldots, i_a}]$ the $(a-1)$-tensor field that arises from $Repla^{i,t}_{\phi_{u+1},\omega}[C_g^{l,i_1 \ldots, i_a}]$ by switching ϕ_{u+1} and ω.

For each $i_h \in I^+_l$ and each $t \in Set(T(i))$, we denote by $Repla^{i,t}_{\phi_{u+1},\omega}[C_g^{l,i_1 \ldots, i_a}]$ the $(a-1)$ tensor field that arises from $C_g^{l,i_1 \ldots i_a}$ by erasing the index ${}_t$ and making the index ${}_i$ contract against a factor $\nabla\phi_{u+1}$. We also make the index ^{t} contract against a factor $\nabla\omega$. We again denote by $Repla^{i,t}_{\omega,\phi_{u+1}}[C_g^{l,i_1 \ldots i_a}]$ the $(a-1)$-tensor field that arises from $Repla^{i,t}_{\phi_{u+1},\omega}[C_g^{l,i_1 \ldots i_a}]$ by switching ϕ_{u+1} and ω.

We then calculate that for each $l \in L_\mu$:

---

[115] Recall that I_l stands for the set of free indices in the tensor field $C_g^{l,i_1 \ldots i_a}$.

[116] Recall that $L = L_\mu \bigcup L_{>\mu}$ is the index set of the tensor fields $C_g^{l,i_1 \ldots i_a}$ in our Lemma hypothesis (1.6).

[117] Recall that I_l stands for the set of free indices in $C^{l,i_1 \ldots i_a}$.

[118] Observe that the indices that belong to I^+_l will be the index ${}_j$ in some factor $\nabla^{(\nu)} R_{ijkl}$ that has arisen from a de-symmetrization as above.

[119] Recall that $T(i)$ stands for the factor in $C_g^{l,i_1 \ldots i_a}$ to which the index ${}_i$ belongs.

$$Sub_{\omega}^{\sigma+u+2,\beta}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots,i_a}_g]\} = \sum_{i_h\in I^*_l\bigcup I^+_l}\sum_{t\in Set[T(i)]}$$
$$Xdiv_{i_1}\dots X\hat{d}iv_{i_h}\dots Xdiv_{i_a}\{Repla^{i,t}_{\phi_{u+1},\omega}[C^{l,i_1\dots,i_a}_g]+Repla^{t,i}_{\phi_{u+1},\omega}[C^{l,i_1\dots,i_a}_g]\}. \tag{7.55}$$

Convention: For future reference, we will further subdivide the index sets I^*_l, I^+_l: If there is a unique selected factor we define $I^{*,1}_l = I^*_l\bigcap I_1$ and analogously $I^{*,2}_l = I^*_l\bigcap I_2$ and similarly for I^+_l. (Recall that I_1 (or I_2) stand for the sets of free indices that belong (do not belong) to the selected factor, when the selected factor is unique). If there are multiple selected factors $\{T_i\}_{i=1}^{b_l}$, we define $I^{*,1,T_i}_l = I^*_l\bigcap I^{T_i}_1$ and analogously $I^{*,2,T_i}_l = I^*_l\bigcap I^{T_i}_2$. (Recall that $I^{T_i}_1$ is the set of free indices that belong to the selected factor T_i and $I^{T_i}_2$ is the set of free indices that do not belong to the selected factor T_i).

Analogously, we deduce that for each $l\in L\setminus L_\mu$:

$$Sub_{\omega}^{\sigma+u+2,\beta}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots,i_a}_g]\} = \sum_{m\in M\bigcup M^\sharp} a_m$$
$$Xdiv_{i_1}\dots Xdiv_{i_a}C^{u,i_1\dots,i_a,i_{a+1},i_{a+2}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{a+1}}\phi_{u+1}\nabla_{i_{a+2}}\omega. \tag{7.56}$$

Finally, we seek to understand $Sub_{\omega}^{\sigma+u+2,\gamma}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots,i_a}_g]\}$.

Definition 7.8 *For each $C^{l,i_1\dots i_a}_g$, we denote by I^d_l the set of pairs of free indices that belong to the same factor, such that at least one of them is a derivative index.*

Now, for each $l\in L$ and each $({}_{i_k},{}_{i_l})\in I^d_l$, we assume with no loss of generality that ${}_{i_k}$ is a derivative index. We also denote by $\{F_1,\dots,F_{\sigma-1}\}$ the set of real factors (i.e. factors that are not in the form $\nabla\phi_h$) in $C^{l,i_1\dots,i_a}_g$ *other than* the factor to which ${}_{i_k},{}_{i_l}$ belong. We then denote by $Re^{K,\phi_{u+1},\omega}_{i_k,i_l}[C^{l,i_1\dots,i_a}_g]$ the $(a-1)$-tensor field that arises from $C^{l,i_1\dots,i_a}_g$ by erasing ${}_{i_k}$, contracting ${}_{i_l}$ against a factor $\nabla\phi_{u+1}$ and then hitting the factor F_K by a derivative ∇_z and contracting ${}_z$ against a factor $\nabla^z\omega$. We denote by $Re^{K,\omega,\phi_{u+1}}_{i_k,i_l}[C^{l,i_1\dots,i_a}_g]$ the $(a-1)$-tensor field that arises from $Re^{K,\omega,\phi_{u+1}}_{i_k,i_l}[C^{l,i_1\dots,i_a}_g]$ by switching ϕ_{u+1} and ω. We then calculate that for each $l\in L$:

$$Sub_{\omega}^{\sigma+u+2,\gamma}\{Image^{1,\beta}_{\phi_{u+1}}[Xdiv_{i_1}\dots Xdiv_{i_a}C^{l,i_1\dots,i_a}_g]\} =$$
$$\sum_{(i_k,i_l)\in I^d_l}\sum_{K=1}^{\sigma-1} Xdiv_{i_1}\dots X\hat{d}iv_{i_k}\dots X\hat{d}iv_{i_l}\dots Xdiv_{i_a} \tag{7.57}$$
$$\{Re^{K,\omega,\phi_{u+1}}_{i_k,i_l}[C^{l,i_1\dots,i_a}_g]+Re^{K,\phi_{u+1},\omega}_{i_k,i_l}[C^{l,i_1\dots,i_a}_g]\}.$$

In conclusion, we have shown that:

$$
\begin{aligned}
&Sub_{\omega}^{\sigma+u+2}\{Image^{1,\beta}_{\phi_{u+1}}L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\}=\\
&\sum_{l\in L}a_l\sum_{(i_k,i_l)\in(I\times I)^\sharp}Xdiv_{i_1}\dots X\hat{div}_{i_k}\dots X\hat{div}_{i_l}\dots Xdiv_{i_a}C^{l,i_1\dots,i_a}_g\nabla_{i_k}\phi_{u+1}\nabla_{i_l}\omega+\\
&\sum_{l\in L_\mu}a_l\sum_{i_h\in I^*_l\bigcup I^+_l}\sum_{t\in T(i_h)}Xdiv_{i_1}\dots X\hat{div}_{i_h}\dots Xdiv_{i_a}\\
&\{Repla^{i,t}_{\phi_{u+1},\omega}[C^{l,i_1\dots,i_a}_g]+Repla^{t,i}_{\phi_{u+1},\omega}[C^{l,i_1\dots,i_a}_g]\}\\
&+\sum_{m\in M\bigcup M^\sharp\bigcup M^{\sharp\sharp}}a_mC^{m,i_1\dots i_{a+2}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{a+1}}\phi_{u+1}\nabla_{i_{a+2}}\omega+\\
&\sum_{l\in L}a_l\sum_{(i_k,i_l)\in I^d_l}\sum_{K=1}^{\sigma-1}Xdiv_{i_1}\dots X\hat{div}_{i_k}\dots X\hat{div}_{i_l}\dots Xdiv_{i_a}\\
&\{Re^{K,\omega,\phi_{u+1}}_{i_k,i_l}[C^{l,i_1\dots,i_a}_g]+Re^{K,\phi_{u+1},\omega}_{i_k,i_l}[C^{l,i_1\dots,i_a}_g]\}\\
&+\sum_{j\in J^{\sigma+u+2}}a_jC^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega).
\end{aligned}
\tag{7.58}
$$

Replacing the above into (7.47) we obtain a new equation, after all this extensive analysis of the equation $Image^{1,\beta}_{\phi_{u+1}}[L_g]=0$:

$$
\begin{aligned}
&\sum_{l\in L} a_l \sum_{(i_k,i_l)\in(I\times I)^\sharp} Xdiv_{i_1}\dots X\hat{d}iv_{i_k}\dots X\hat{d}iv_{i_l}\dots Xdiv_{i_a} C^{l,i_1\dots,i_a}_g \nabla_{i_k}\phi_{u+1}\nabla_{i_l}\omega\\
&+\sum_{l\in L_\mu} a_l \sum_{i_h\in I^*_l\bigcup I^+_l} \sum_{t\in T(i_h)} Xdiv_{i_1}\dots X\hat{d}iv_{i_h}\dots Xdiv_{i_a}\\
&\{Repla^{i,t}_{\phi_{u+1},\omega}[C^{l,i_1\dots,i_a}_g] + Repla^{t,i}_{\phi_{u+1},\omega}[C^{l,i_1\dots,i_a}_g]\}\\
&+\sum_{l\in L_\mu} a_l \sum_{i_f\in I^\sharp} Xdiv_{i_1}\dots X\hat{d}iv_{i_f}\dots Xdiv_{i_\mu}\\
&\{FRep^{i_h,1,\omega,\phi_{u+1}}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\\
&+FRep^{i_h,1,\phi_{u+1},\omega}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\\
&+FRep^{i_h,2,\omega,\phi_{u+1}}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+\\
&FRep^{i_h,2,\phi_{u+1},\omega}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}+\\
&(\sum_{m\in M_{\mu-1}} a_m Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} C^{m,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_\mu}\phi_{u+1}\nabla_{i_{\mu+1}}\omega)\\
&+\sum_{m\in M\bigcup M^\sharp\bigcup M^{\sharp\sharp}} a_m C^{m,i_1\dots i_{a+2}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{a+1}}\phi_{u+1}\nabla_{i_{a+2}}\omega\\
&+\sum_{l\in L} a_l \sum_{(i_k,i_l)\in I^d_l} \sum_{K=1}^{\sigma-1} Xdiv_{i_1}\dots X\hat{d}iv_{i_k}\dots X\hat{d}iv_{i_l}\dots Xdiv_{i_a}\\
&\{Re^{K,\omega,\phi_{u+1}}_{i_k,i_l}[C^{l,i_1\dots,i_a}_g] + Re^{K,\phi_{u+1},\omega}_{i_k,i_l}[C^{l,i_1\dots,i_a}_g]\}+\\
&\sum_{\zeta\in Z} a_\zeta Xdiv_{i_1}\dots Xdiv_{\mu-1} Hit^K_\omega[C^{\zeta,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})]+\\
&+\sum_{j\in J^{\sigma+u+2}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)=\\
&\sum_{t\in T^{\sigma+u+2}} a_t C^t_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega),
\end{aligned}
\tag{7.59}
$$

modulo complete contractions of length $\geq \sigma+u+3$. The sublinear combination $\sum_{m\in M_{\mu-1}}\dots$ appears only in the special subcase of case B. The linear combination on the RHS stands for generic notation (see the notational convention introduced after (7.3)).

In fact, we observe that the minimum length of the complete contractions above is $\sigma+u+2$, and that all terms on the LHS have two factors $\nabla\phi_{u+1},\nabla\omega$, while each term on the RHS has at least one term $\nabla^{(A)}\phi_{u+1}$ or $\nabla^{(A)}\omega$, with $A\geq 2$.

Therefore, since the above holds formally, we derive:

$$\sum_{l\in L} a_l \sum_{(i_k,i_l)\in(I\times I)^\sharp} Xdiv_{i_1}\dots X\hat{div}_{i_k}\dots X\hat{div}_{i_l}\dots Xdiv_{i_a} C^{l,i_1\dots,i_a}_g \nabla_{i_k}\phi_{u+1}\nabla_{i_l}\omega+$$
$$\sum_{l\in L_\mu} a_l \sum_{i_h\in I^*_l\bigcup I^+_l}\sum_{t\in T(i_h)} Xdiv_{i_1}\dots X\hat{div}_{i_h}\dots Xdiv_{i_a}$$
$$\{Repla^{i,t}_{\phi_{u+1},\omega}[C^{l,i_1\dots,i_a}_g] + Repla^{t,i}_{\phi_{u+1},\omega}[C^{l,i_1\dots,i_a}_g]\}$$
$$+\sum_{l\in L_\mu} a_l \sum_{i_f\in I^\sharp} Xdiv_{i_1}\dots X\hat{div}_{i_f}\dots Xdiv_{i_\mu}$$
$$\{FRep^{i_h,1,\omega,\phi_{u+1}}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$
$$+FRep^{i_h,1,\phi_{u+1},\omega}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$
$$+FRep^{i_h,2,\omega,\phi_{u+1}}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+$$
$$FRep^{i_h,2,\phi_{u+1},\omega}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}+$$
$$(\sum_{m\in M_{\mu-1}} a_m Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} C^{m,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_\mu}\phi_{u+1}\nabla_{i_{\mu+1}}\omega)$$
$$+\sum_{m\in M\bigcup M^\sharp\bigcup M^{\sharp\sharp}} a_m C^{m,i_1\dots i_{a+2}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{a+1}}\phi_{u+1}\nabla_{i_{a+2}}\omega$$
$$+\sum_{l\in L} a_l \sum_{(i_k,i_l)\in I^d_l}\sum_{K=1}^{\sigma-1} Xdiv_{i_1}\dots X\hat{div}_{i_k}\dots X\hat{div}_{i_l}\dots Xdiv_{i_a}$$
$$\{Re^{K,\omega,\phi_{u+1}}_{i_k,i_l}[C^{l,i_1\dots,i_a}_g] + Re^{K,\phi_{u+1},\omega}_{i_k,i_l}[C^{l,i_1\dots,i_a}_g]\}+$$
$$\sum_{\zeta\in Z} a_\zeta Xdiv_{i_1}\dots Xdiv_{\mu-1} Hit^K_\omega[C^{\zeta,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})]+$$
$$+\sum_{j\in J^{\sigma+u+2}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)=0, \tag{7.60}$$

modulo complete contractions of length $\geq \sigma+u+3$.

We denote the above equation by:

$$Im^{1,\beta}_{\phi_{u+1}}[L_g]=0, \tag{7.61}$$

for short. We repeat that the contractions appearing in the above equation all have length $\sigma+u+2$, and the equation holds modulo complete contractions of length $\geq \sigma+u+3$.

The operation ***Soph*****:** We now define a formal operation *Soph* that acts on the complete contractions above: For each $C_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)$ (with factors $\nabla\phi_{u+1}$, $\nabla\omega$ which are necessarily contracting against different factors), we first replace the two factors $\nabla_i\phi_{u+1},\nabla_j\omega$ by a factor g_{ij}. Then, we add a derivative index ∇_u onto the selected factor and contract it against

a factor $\nabla^u \phi_{u+1}$ (if there are multiple selected factors we perform the same operation for each of them and then add). Finally, we multiply the complete contraction by a factor $\frac{1}{2}$. This definition extends to tensor fields and linear combinations.

Observe that when this operation acts on the complete contractions in $Im^{1,\beta}_{\phi_{u+1}}[L_g]$, it produces complete contractions of length $\sigma+u+1$ with a factor $\nabla\phi_{u+1}$ and with a weak character $Weak(\vec{\kappa}^+_{simp})$.

Observe that since (7.61) holds formally, it follows that:

$$Soph\{Im^{1,\beta}[L_g]\} = 0, \tag{7.62}$$

modulo complete contractions of length $\geq \sigma + u + 2$.

7.4 Preparation for the grand conclusion.

Schematically, our goal for the rest of this section will be to *add* (7.62) to the equation (6.24) (or (6.25), (6.26), depending on the form of the selected factor), thus deriving a new true equation which we denote by:

$$Image^{1,+}_{\phi_{u+1}}[L_g] + Soph\{Im^{1,\beta}[L_g]\} = 0; \tag{7.63}$$

this holds modulo complete contractions of length $\geq \sigma + u + 2$. *This* new true equation is the "grand conclusion", which is the main aim of our present paper. The "grand conclusion" will almost directly imply Lemma 1.3 in case A. It will also be the main tool in deriving Lemma 1.3 in case B–this will be done in section 9.

A few easy observations:

$$\begin{gathered}\sum_{j\in J^{\sigma+u+2}} a_j Soph[C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)] = \\ \sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1}).\end{gathered} \tag{7.64}$$

We also observe that for $m \in M$,

$$Soph\{Xdiv_{i_1}\dots Xdiv_{i_a} C^{m,i_1\dots i_{a+2}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{a+1}}\phi_{u+1}\nabla_{i_{a+2}}\omega\} \tag{7.65}$$

is an acceptable contributor (see Definition 5.1), while if $m \in M^\sharp$ (7.65 is an un-acceptable contributor with one un-acceptable factor, and for $m \in M^{\sharp\sharp}$ (7.65) is a linear combination of terms with all the properties of contributors, *but* there will be *two* un-acceptable factors $\nabla\Omega, \nabla\Omega_{h'}$ that are contracting against each other (and if $a = \mu$ then all free indices are non-special). We have denoted by

$$\sum_{h\in H} a_h Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$$

a generic linear combination of contributors (acceptable or with one unacceptable factor $\nabla\Omega_h$ as in the conclusion of Lemma 1.3); we also denote by

$$\sum_{h\in H^{\S\S}} a_h Xdiv_{i_1}\dots Xdiv_{i_a} C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})$$

generic linear combinations of terms line the ones indexed in $M^{\sharp\sharp}$.

By definition, we observe that

$$Soph\{\sum_{\zeta\in Z} a_\zeta Xdiv_{i_1}\dots Xdiv_{a_{\mu-1}}\sum_{K=1}^{\sigma-1} Hit^K_\omega[C^{\zeta,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})]\} \tag{7.66}$$

is a contributor,[120] because acting by $Soph\{\dots\}$ on the operation $\sum_{K=1}^{\sigma-1} Hit^K_\omega$ gives rise to another $Xdiv$ (see the Definition 7.5 and the discussion under it).

Now, we proceed to derive some delicate cancellations occurring in (7.63).

Observe that:

$$\begin{aligned}&Soph\{\sum_{l\in L} a_l \sum_{(i_k,i_l)\in(I\times I)^\sharp} Xdiv_{i_1}\dots X\hat{d}iv_{i_k}\dots X\hat{d}iv_{i_l}\dots Xdiv_{i_a} C^{l,i_1\dots,i_a}_g \nabla_{i_k}\phi_{u+1}\nabla_{i_l}\omega\}\\ &=\sum_{i=1}^{b_l}\{\sum_{(i_k,i_l)\in(I_2^{T_i})^{2,dif}} Xdiv_{i_1}\dots X\hat{d}iv_{i_k}\dots X\hat{d}iv_{i_l}\dots Xdiv_{i_a}[C^{l,i_1\dots i_a,i_*|T_i}_g\\ &(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)g^{i_k i_l}]\nabla_{i_*}\phi_{u+1}+\sum_{i_k\in I_1^{T_i},i_l\in I_2^{T_i}} Xdiv_{i_1}\dots X\hat{d}iv_{i_k}\dots X\hat{d}iv_{i_l}\\ &\dots Xdiv_{i_a}\nabla^{i_*}_{T_i}[C^{l,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)g^{i_k i_l}]\nabla_{i_*}\phi_{u+1}\}.\end{aligned} \tag{7.67}$$

For our next observation, we will look at each $l\in L$ and pick out each pair of indices $({}_{i_k},{}_{i_l})\in I^d_l$. We assume with no loss of generality that ${}_{i_k}$ is a derivative index (recall that I^d_l stands for the set of pairs of indices that belong to the same factor in $C^{l,i_1\dots i_a}_g$ and at least one of which is a derivative index). Then, for each such pair $({}_{i_k},{}_{i_l})$, we denote by

$$\dot{C}^{l,i_1\dots\hat{i}_k\dots i_\mu,i_*|T_i}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}$$

the tensor field that arises from $C^{l,i_1\dots i_\mu}$ by erasing the index ${}_{i_k}$ and adding a derivative index ∇_{i_*} onto the selected factor T_i and then contracting ∇_{i_*} against a factor $\nabla^{i_*}\phi_{u+1}$. We see that for each $l\in L$:

[120] See Definition 5.1.

$$Soph\{\sum_{l\in L}a_l\sum_{(i_k,i_l)\in I^d_l}\sum_{K=1}^{\sigma-1}Xdiv_{i_1}\dots X\hat{d}iv_{i_k}\dots X\hat{d}iv_{i_l}\dots Xdiv_{i_a}$$

$$\{Re^{K,\omega,\phi_{u+1}}_{i_k,i_l}[C^{l,i_1\dots,i_a}_g]+Re^{K,\phi_{u+1},\omega}_{i_k,i_l}[C^{l,i_1\dots,i_a}_g]\}\}=\sum_{i=1}^{b_l}\sum_{(i_k,i_l)\in I^d_l}$$

$$Xdiv_{i_1}\dots X\hat{d}iv_{i_k}\dots Xdiv_{i_a}\dot{C}^{l,i_1\dots\hat{i}_k\dots i_\mu,i_*|T_i}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}. \tag{7.68}$$

Furthermore, for each selected factor T_i, let us denote by $I^{d,non-T_i}_l\subset I^d_l$ the subset of I^d_l that consists of pairs of free indices that do not belong to the selected factor T_i. We observe:

$$\sum_{i=1}^{b_l}\sum_{i_y\in I^{T_i}_2}\sigma(i_y)Xdiv_{i_1}\dots X\hat{d}iv_{i_y}\dots Xdiv_{i_\mu}[\nabla^{i_y}_{T_i}C^{l,i_1\dots\hat{i}_y\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$$

$$\nabla_{i_y}\phi_{u+1}]=\sum_{i=1}^{b_l}Soph\{\sum_{i_h\in I^{*,2|T_i}_l\bigcup I^{+,2|T_i}_l}\sum_{t\in Set[T(i)]}Xdiv_{i_1}\dots X\hat{d}iv_{i_h}\dots Xdiv_{i_a}$$

$$\{Repla^{i,t}_{\phi_{u+1},\omega}[C^{l,i_1\dots,i_a}_g]+Repla^{t,i}_{\phi_{u+1},\omega}[C^{l,i_1\dots,i_a}_g]\}+2\sum_{i=1}^{b_l}\sum_{(i_k,i_l)\in I^{d,non-T_i}_l}$$

$$Xdiv_{i_1}\dots X\hat{d}iv_{i_k}\dots Xdiv_{i_a}\dot{C}^{l,i_1\dots\hat{i}_k\dots i_\mu,i_*|T_i}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}\}. \tag{7.69}$$

Now, a few more delicate observations. For each $l\in L_\mu$ and each selected factor T_i, we denote by $I^{\sharp,T_i}_2\subset I^\sharp$[121] the index set of the indices that belong to a factor $S_*\nabla^{(\nu)}R_{ijkl}\neq T_i$. We also denote by $I^{\sharp,T_i}_1=I^\sharp\setminus I^{\sharp,T_i}_2$. Furthermore, for each tensor field $C^{l,i_1\dots,i_\mu}_g$ and each free index ${}_{i_h}$ in that tensor field, we will set $2_{i_h}=2$ if ${}_{i_h}$ belongs to a factor $\nabla^{(m)}R_{ijkl}$ or $S_*\nabla^{(\nu)}R_{ijkl}$ and $2_{i_h}=0$ if it belongs to a factor $\nabla^{(A)}\Omega_h$. Then, comparing the discussion above (6.16) and (7.34), we derive that:

[121] Recall that $I^\sharp$ stands for the set of free indices in $C^{l,i_1\dots i_a}_g$ that belong to a factor $S_*\nabla^{(\nu)}R_{ijkl}$.

$$\sum_{i=1}^{b_l}\sum_{i_y\in I_2^{T_i}}\tau(i_y)Xdiv_{i_1}\dots X\hat{div}_{i_y}\dots Xdiv_{i_\mu}[\nabla^{i_y}_{T_i}C^{l,i_1\dots\hat{i_y}\dots i_\mu}_g(\Omega_1,\dots,\Omega_p)\nabla_{i_y}\phi_{u+1}]$$
$$+Soph\{\sum_{i_h\in I_2^{\sharp,T_i}}Xdiv_{i_1}\dots X\hat{div}_{i_h}\dots Xdiv_{i_a}$$
$$\{FRep^{i_h,1,\omega,\phi_{u+1}}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$
$$+FRep^{i_h,1,\phi_{u+1},\omega}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$
$$+FRep^{i_h,2,\omega,\phi_{u+1}}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$
$$+FRep^{i_h,2,\phi_{u+1},\omega}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}\}=$$
$$\sum_{i=1}^{b_l}\sum_{i_y\in I_2^{T_i}}2_{i_y}Xdiv_{i_1}\dots X\hat{div}_{i_y}\dots Xdiv_{i_\mu}[\nabla^{i_y}_{T_i}C^{l,i_1\dots\hat{i_y}\dots i_\mu}_g(\Omega_1,\dots,\Omega_p)\nabla_{i_y}\phi_{u+1}]. \tag{7.70}$$

On the other hand, we observe that:

$$Soph\{\sum_{m\in M_{\mu-1}}a_mXdiv_{i_1}\dots Xdiv_{i_{\mu-1}}C^{m,i_1\dots,i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)\}$$
$$=\sum_{b\in B'}a_bXdiv_{i_1}\dots Xdiv_{i_{\mu-1}}C^{b,i_1\dots,i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}). \tag{7.71}$$

(Recall that the sublinear combination indexed in $M_{\mu-1}$ appears only in the special subcase of case B). The linear combination indexed in B' is a generic linear combination defined in Definition 5.2.

Next, we note some further cancellations, for each $C^{l,i_1\dots,i_\mu}_g$ with at least one free index in the selected factor. In the case where the selected factor is of the form $\nabla^{(A)}\Omega_h$ (in which case it is unique) we must have:

$$Soph\{\sum_{i_h\in I_l^{*,1}}\sum_{t\in Set(T(i_h))}Xdiv_{i_1}\dots X\hat{div}_{i_h}\dots Xdiv_{i_a}$$
$$\{Repla^{i_h,t}_{\phi_{u+1},\omega}[C^{l,i_1\dots,i_a}_g]+Repla^{t,i_h}_{\phi_{u+1},\omega}[C^{l,i_1\dots,i_a}_g]\}\}$$
$$=\sum_{i_h\in I_1}A^\sharp Xdiv_{i_1}\dots X\hat{div}_{i_h}\dots Xdiv_{i_\mu}C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_h}\phi_{u+1}; \tag{7.72}$$

(Recall that $A^\sharp$ stands for the number of indices in the factor $\nabla^{(A)}\Omega_h$ that are not free and not contracting against a factor $\nabla\phi_h$). On the other hand, if the selected factor(s) is (are) of the form $\nabla^{(m)}R_{ijkl}$ we will have:

$$Soph\{\sum_{i_h\in I_l^{*,1}}\sum_{t\in Set(T(i_h))} Xdiv_{i_1}\dots X\hat{d}iv_{i_h}\dots Xdiv_{i_a}$$

$$\{Repla^{i_h,t}_{\phi_{u+1},\omega}[C^{l,i_1\dots,i_a}_g]+Repla^{t,i_h}_{\phi_{u+1},\omega}[C^{l,i_1\dots,i_a}_g]\}\}=\sum_{i=1}^{b_l}\sum_{i_h\in I_1^{T_i}}(m_i^\sharp+4) \tag{7.73}$$

$$Xdiv_{i_1}\dots X\hat{d}iv_{i_h}\dots Xdiv_{i_\mu}C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_h}\phi_{u+1};$$

(recall that $m_i^\sharp$ stands for the number of derivative indices in the selected factor $T_i=\nabla^{(m)}R_{ijkl}$ that are not free and not contracting against a factor $\nabla\phi_h$).

Finally, in the case where the selected factor is of the form $S_*\nabla^{(\nu)}R_{ijkl}$ (in which case it is again unique), for each $l\in L_\mu$ with at least one free index in the selected factor, we find:

$$Soph\{\sum_{i_h\in I_l^{*,1}\bigcup I_1^{+,1}}\sum_{t\in Set(T(i_h))} Xdiv_{i_1}\dots X\hat{d}iv_{i_h}\dots Xdiv_{i_a}$$
$$\{Repla^{i_h,t}_{\phi_{u+1},\omega}[C^{l,i_1\dots,i_a}_g]+Repla^{t,i_h}_{\phi_{u+1},\omega}[C^{l,i_1\dots,i_a}_g]\}\}+Soph\{\sum_{i_f\in I_1^\sharp}Xdiv_{i_1}\dots$$
$$X\hat{d}iv_{i_h}\dots Xdiv_{i_a}\{FRep^{i_h,1,\omega,\phi_{u+1}}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$
$$+FRep^{i_h,1,\phi_{u+1},\omega}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$$
$$+FRep^{i_h,2,\omega,\phi_{u+1}}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]+$$
$$FRep^{i_h,2,\phi_{u+1},\omega}[C^{l,i_1\dots i_a|f(i_h)}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\}\}=$$
$$\sum_{i_h\in I_1}(\nu^\sharp+2)Xdiv_{i_1}\dots X\hat{d}iv_{i_h}\dots Xdiv_{i_\mu}C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_h}\phi_{u+1}; \tag{7.74}$$

(recall that $\nu^\sharp$ stands for the number of indices ${}_{r_1},\dots,{}_{r_\nu},{}_j$ in the selected factor $S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ijkl}$ that are not free and not contracting against a factor $\nabla\phi_h$).

8 The grand conclusion, and the proof of Lemma 1.3.

8.1 The grand conclusion.

Now, we combine all the cancellations we have noted in the previous subsection to derive the "grand conclusion". When the selected factor(s) is (are) of the form $S_*\nabla^{(\nu)}R_{ijkl}$ or $\nabla^{(m)}R_{ijkl}$, the grand conclusion will be the equation:

$$Image^{1,+}_{\phi_{u+1}}[L_g] + Soph\{Im^{1,\beta}_{\phi_{u+1}}[L_g]\} + \{L_g(\Omega_1 \cdot \phi_{u+1}, \dots, \Omega_p, \phi_1, \dots, \phi_u)$$
$$+ \dots + L_g(\Omega_1, \dots, \Omega_X \cdot \phi_{u+1}, \dots, \Omega_p, \phi_1, \dots, \phi_u)\} = 0. \tag{8.1}$$

(Recall that $\Omega_1, \dots, \Omega_X$ are the factors in $\vec{\kappa}_{simp}$ that are not contracting against any factor $\nabla\phi_h$. The terms in $\{\dots\}$ appear *only* when the selected factor(s) is (are) curvature terms).

When the selected factor is of the form $\nabla^{(B)}\Omega_x$, the grand conclusion will be the equation:

$$Image^{1,+}_{\phi_{u+1}}[L_g] + Soph\{Im^{1,\beta}_{\phi_{u+1}}[L_g]\} = 0. \tag{8.2}$$

For future reference, we put down a few facts before we write out the "grand conclusion":

Recall notation: Recall that s stands for the total number of factors $\nabla^{(m)}R_{ijkl}$, $S_*\nabla^{(\nu)}R_{ijkl}$ in the simple character $\vec{\kappa}_{simp}$ (all the tensor fields in (1.6) have this given simple character–see the introduction of the present paper for a simplified discussion of this notion). Recall that for each $C^{l,i_1\dots i_\mu}_g$, $l \in L_\mu$: γ (or γ_i if there are multiple selected factors T_i) stands for the number of indices in $C^{l,i_1\dots i_\mu}_g$ that do not belong to the selected factor and are not contracting against a factor $\nabla\phi_h$. We also recall that I_1 (or $I_1^{T_i}$ if there are multiple selected factors) stands for the set of free indices that belong to the selected factor, and I_2 (or $I_2^{T_i}$ if there are multiple selected factors) stands for the set of free indices that *do not* belong to the selected factor. We also recall that for each $l \in L_\mu$ and each free index ${}_{i_h} \in I_2$ (or ${}_{i_h} \in I_2^{T_i}$) which belongs to $C^{l,i_1\dots i_\mu}_g$, 2_{i_h} stands for the number 2 if the free index ${}_{i_h}$ belongs to a factor of the from $\nabla^{(m)}R_{ijkl}$ or $S_*\nabla^{(\nu)}R_{ijkl}$, and it will be zero if it belongs to a factor of the form $\nabla^{(B)}\Omega_x$. Now, we define $\overline{2}_{i_h}$ to equal number 2 if the free index ${}_{i_h}$ belongs to a factor of the from $\nabla^{(m)}R_{ijkl}$ or $S_*\nabla^{(\nu)}R_{ijkl}$, and to equal 1 if it belongs to a factor of the form $\nabla^{(B)}\Omega_x$.

Finally, we recall: When the selected factor is of the form $S_*\nabla^{(\nu)}R_{ijkl}$ then (for each μ-tensor field $C^{l,i_1\dots i_\mu}_g$, $l \in L_\mu$) $\nu^\sharp$ stands for the number of indices in the selected factor $S_*\nabla^{(\nu)}R_{ijkl}$ that are not free and not contacting against a factor $\nabla\phi_h$. When the selected factor(s) is (are) of the form $\nabla^{(m)}R_{ijkl}$, then (for each μ-tensor field $C^{l,i_1\dots i_\mu}_g$, $l \in L_\mu$ and) for each selected factor $\nabla^{(m_i)}R_{ijkl}$, $m_i^\sharp$ stands for the number of derivative indices that are not free and not contracting against a factor $\nabla\phi_h$. Lastly, when the selected factor is of the form $\nabla^{(A)}\Omega_k$ then (for each μ-tensor field $C^{l,i_1\dots i_\mu}_g$, $l \in L_\mu$) $A^\sharp$ stands for the number of indices in $\nabla^{(A)}\Omega_k$ that are not free and not contracting against a factor $\nabla\phi_h$.

Then, if the selected factor is of the form $S_*\nabla^{(\nu)}R_{ijkl}$, the "grand conclusion" may be written in the form:

$$\sum_{l\in L_\mu} a_l\{-\sum_{i_h\in I_1}(\gamma+(|I_1|-1)+\nu^\sharp-2(s-1)-X)Xdiv_{i_1}\dots X\hat{d}iv_{i_h}Xdiv_{i_\mu}$$
$$C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_h}\phi_{u+1}+\sum_{i_h\in I_2}\overline{2}_{i_h}$$
$$Xdiv_{i_1}\dots X\hat{d}iv_{i_h}\dots Xdiv_{i_\mu}\nabla^{i_*}_{sel}[C^{l,i_1\dots\hat{i}_h\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\nabla_{i_*}\phi_{u+1}$$
$$-\{\sum_{(i_k,i_l)\in I^{d,non-sel}_l} Xdiv_{i_1}\dots X\hat{d}iv_{i_k}\dots Xdiv_{i_\mu}\dot{C}^{l,i_1\dots\hat{i}_k\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$$
$$\nabla_{i_*}\phi_{u+1}+\sum_{(i_k,i_l)\in I^{d,sel}_l}\sum_{S=1}^{\sigma-1}Xdiv_{i_1}\dots X\hat{d}iv_{i_k}\dots X\hat{d}iv_{i_l}\dots Xdiv_{i_\mu}Xdiv_{i_z}$$
$$\tilde{C}^{l,i_1\dots,\hat{i}_k\dots,i_\mu,i_z|S}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_l}\phi_{u+1}\}+$$
$$\sum_{j\in J^{\sigma+u+2}} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})+$$
$$(\sum_{b\in B'} a_b Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}}C^{b,i_1\dots,i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}))+$$
$$\sum_{h\in H\bigcup H^{\S\S}} a_h Xdiv_{i_1}\dots Xdiv_{i_a}C^{h,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}=0, \tag{8.3}$$

modulo complete contractions of length $\geq \sigma+u+2$. $\sum_{b\in B'}\dots$ (defined in Definition 5.2) appears *only* in the special subcase of case B.

If the selected factor(s) is (are) of the from $\nabla^{(m)}R_{ijkl}$, the "grand conclusion" we obtain is very analogous:

$$
\begin{aligned}
&\sum_{l\in L_\mu} a_l \sum_{i=1}^{b_l}\{-\sum_{i_h\in I_1^{T_i}}(\gamma_i+(|I_1^{T_i}|-1)+m_i^\sharp-2(s-1)-X)\\
&Xdiv_{i_1}\dots X\hat{div}_{i_h}\dots Xdiv_{i_\mu}C_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_h}\phi_{u+1}+\\
&\sum_{i_h\in I_2^{T_i}}\overline{2}_{i_h}\nabla^{i_*}_{T_i}[C_g^{l,i_1\dots\hat{i}_h\dots i_\mu,i_*}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\nabla_{i_*}\phi_{u+1}-\sum_{(i_k,i_l)\in I_l^{d,non-T_i}}\\
&Xdiv_{i_1}\dots X\hat{div}_{i_k}\dots Xdiv_{i_a}\dot{C}_g^{l,i_1\dots\hat{i}_k\dots i_\mu,i_*}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}\\
&+\sum_{(i_k,i_l)\in I_l^{d,T_i}}\sum_{S=1}^{\sigma-1}Xdiv_{i_1}\dots X\hat{div}_{i_k}\dots X\hat{div}_{i_l}\dots Xdiv_{i_\mu}Xdiv_{i_z}\\
&\tilde{C}_g^{l,i_1\dots,\hat{i}_k\dots,i_\mu,i_z|S}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_l}\phi_{u+1}\}\\
&+\sum_{j\in J^{\sigma+u+2}}a_jC_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})+\\
&(\sum_{b\in B'}a_bXdiv_{i_1}\dots Xdiv_{i_{\mu-1}}C_g^{b,i_1\dots,i_{\mu-1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}))+\\
&\sum_{h\in H\bigcup H^{\S\S}}a_hXdiv_{i_1}\dots Xdiv_{i_a}C_g^{h,i_1\dots i_a,i_*}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}=0,
\end{aligned}
\tag{8.4}
$$

modulo complete contractions of length $\geq \sigma+u+2$. $\sum_{b\in B'}\dots$ (defined in Definition 5.2) appears *only* in the special subcase of case B.

Finally, in the case where the selected factor is of the form $\nabla^{(A)}\Omega_1$, (8.2) gives us the "grand conclusion":

$$\sum_{l\in L_\mu} a_l\{-\sum_{i_h\in I_1}(\gamma+(|I_1|-1)+A^\sharp-2s)Xdiv_{i_1}\dots X\hat{d}iv_{i_h}\dots Xdiv_{i_\mu}C^{l,i_1\dots i_\mu}_g$$
$$(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_h}\phi_{u+1}+$$
$$\sum_{i_h\in I_2} 2_{i_h}\nabla^{i_*}_{sel}[C^{l,i_1\dots\hat{i}_h\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\nabla_{i_*}\phi_{u+1}-\sum_{(i_k,i_l)\in I^{d,non-sel}_l}$$
$$\{Xdiv_{i_1}\dots X\hat{d}iv_{i_k}\dots Xdiv_{i_\mu}\dot{C}^{l,i_1\dots\hat{i}_k\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}$$
$$+\sum_{(i_k,i_l)\in I^{d,sel}_l}\sum_{S=1}^{\sigma-1}Xdiv_{i_1}\dots X\hat{d}iv_{i_k}\dots X\hat{d}iv_{i_l}\dots Xdiv_{i_\mu}Xdiv_{i_z}$$
$$\tilde{C}^{l,i_1\dots,\hat{i}_k\dots,i_\mu,i_z|S}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_l}\phi_{u+1}\}+$$
$$\sum_{j\in J^{\sigma+u+2}} a_jC^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1})+$$
$$(\sum_{b\in B'} a_bXdiv_{i_1}\dots Xdiv_{i_{\mu-1}}C^{b,i_1\dots,i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}))+$$
$$\sum_{h\in H\bigcup H^{\S\S}} a_hXdiv_{i_1}\dots Xdiv_{i_a}C^{h,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}=0, \tag{8.5}$$

modulo complete contractions of length $\ge\sigma+u+2$. $\sum_{b\in B'}\dots$ (defined in Definition 5.2) appears *only* in the special subcase of case B.

We observe that because of our Lemma assumption that $L^*_\mu=\emptyset$,[122] it follows that all the $(\mu-1)$-tensor fields above are acceptable. Moreover, by construction they each have a $(u+1)$-simple character $\vec{\kappa}^+_{simp}$.

Now, we will show in a "Mini-Appendix" below that using the above, we may write:

$$\begin{aligned}&\sum_{h\in H^{\S\S}} a_hXdiv_{i_1}\dots Xdiv_{i_a}C^{h,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}=\\&\sum_{h\in H} a_hXdiv_{i_1}\dots Xdiv_{i_a}C^{h,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}\\&+\sum_{j\in J^{\sigma+u+2}} a_jC^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1}),\end{aligned} \tag{8.6}$$

using generic notation in the right hand side–the sublinear combination in the left hand side is exactly the one appearing in (8.3), (8.4), (8.5).

In view of (8.6) (which we prove below in the appendix), we may assume that $H^{\S\S}=\emptyset$ in (8.3), (8.4), (8.5), whenever we refer to these equations.

[122] See the notation in the *statement* of Lemma 1.3.

8.2 Proof of Lemma 1.3 in case A.

We pick the selected factor(s) to be the second critical factor(s) (see the statement of Lemma 1.3). Recall that in this case A the second critical factor has at least two free indices.

For convenience, in each sublinear combination

$$(\sum_{(i_k,i_l)\in I_l^{d,T_i}})\sum_{S=1}^{\sigma-1}\tilde{C}_g^{l,i_1\dots,\hat{i}_k\dots,i_\mu,i_z|S}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_l}\phi_{u+1},$$

we will assume that among all the factors $F_1,\dots,F_{\sigma-1}$, the first critical factor(s) is (are) F_1 (or $F_1,\dots,F_a$).[123]

We then claim that among all the $(\mu-1)$-tensor fields in (8.3), (8.4), (8.5) (all of which have a $(u+1)$-simple character $\vec{\kappa}^+_{simp}$), the sublinear combination of *maximal* refined double character will be precisely:

$$\sum_{z\in Z'_{Max}}\sum_{l\in L^z}a_l(\sum_{(i_k,i_l)\in I_l^{d,T_i}})\tilde{C}_g^{l,i_1\dots\hat{i}_k\dots,i_\mu,i_z|1}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_l}\phi_{u+1}, \tag{8.7}$$

in the case where there is only one critical factor, and:

$$\sum_{z\in Z'_{Max}}\sum_{l\in L^z}a_l(\sum_{(i_k,i_l)\in I_l^{d,T_i}})\sum_{S=1}^{a}\tilde{C}_g^{l,i_1\dots\hat{i}_k\dots,i_\mu,i_z|S}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_l}\phi_{u+1}, \tag{8.8}$$

in the case where there are $a>1$ critical factors.

This fact essentially follows just by our definitions: Firstly observe that the tensor fields in the above two equations have $M+1$ free indices in some factor. Now, by definition of the *maximal* refined double character we observe that for each $l\in L_\mu$, each factor in $C_g^{l,i_1\dots i_\mu}$ can have at most M free indices in one of its factors. Hence, each tensor field of rank $\mu-1$ in the above three equations *other than the tensor fields with a tilde sign,* $\tilde{C}$, will again have at most M free indices in any one of its factors (and this is double subsequent to the terms in (8.7), (8.8)).

Moreover, for each $l\in L_\mu\setminus\bigcup_{z\in Z'_{Max}}L^z$, we observe by definition that each tensor field in

$$\sum_{S=1}^{\sigma-1}\tilde{C}_g^{l,i_1\dots,\hat{i}_k\dots,i_\mu,i_z|S}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_l}\phi_{u+1} \tag{8.9}$$

will either have at most M free indices in any given factor or will have $M+1$ free indices in one factor but then its refined double character will be subsequent

[123] The expression $\sum_{(i_k,i_l)\in I_l^{d,T_i}}$ will only be present when the selected (second critical) factor is generic in the form $\nabla^{(m)}R_{ijkl}$.

to $\vec{L^z}, z \in Z'_{Max}$: This second claim just follows by the construction of the tensor fields above: If $l \in L_\mu \setminus \bigcup_{z\in Z'_{Max}} L^z$ then the refined double character of $C_g^{l,i_1\dots i_\mu}$ will be either doubly subsequent or "equipolent" to each refined double character $\vec{L}^z$, $z \in Z'_{Max}$ (which corresponds to the tensor fields $C_g^{l,i_1\dots i_\mu}, l \in L^z, z \in Z'_{Max}$).[124] Now $\tilde{C}_g^{l,i_1\dots,\hat{i}_k\dots,i_\mu,i_z|S}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ formally arises from $C^{l,i_1\dots i_\mu}$ by erasing the free index ${}_{i_k}$ from the (selected) factor T_i and adding a new free index ∇_{i_z} onto another factor, with M free indices. Thus, our claim just follows by the definition of ordering among refined double characters.[125]

So we observe that the "grand conclusion" proves Lemma 1.3 in the case A: The sublinear combination (8.7) in the grand conclusion is precisely the first line in (1.14). All the other $(\mu-1)$-tensor fields in the grand conclusion are in the general form $\sum_{\nu\in N} a_\nu \dots$ described in the claim of Lemma 1.3. Also, the tensor fields indexed in H (with rank $\geq \mu$) are in the same general form as the tensor fields indexed in $T_1 \bigcup T_2 \bigcup T_3 \bigcup T_4$ in (1.14). □

Notes Regarding case B: We will prove Lemma 1.3 in case B in section 9 (and our proof there will heavily rely on the grand conclusion above). We only end this section with a remark, which will be essential in the proof of Lemma 1.3 in case B:

Important Remark: The quantities in parentheses in the first lines of (8.3), (8.4), (8.5) are *universal*, i.e. they only depend on the simple character $\vec{\kappa}_{simp}$, and on the *form* of the selected factor T_d (meaning whether it is of the form $S_*\nabla^{(\nu)}R_{ijkl}$, $\nabla^{(m)}R_{ijkl}$ or $\nabla^{(A)}\Omega_h$): We denote those quantities (inside the parentheses) by q_z. We observe that in the case of (8.5):

$$q_d = n - 2u - \mu - 1. \tag{8.10}$$

In the case of (8.3):

$$q_d = n - 2u - \mu - 1 - X. \tag{8.11}$$

Whereas in the case of (8.4):

$$q_d = n - 2u - \mu - 3 - X. \tag{8.12}$$

(We will define $Q_d = |I_1| \cdot q_d$, for future reference).

8.3 Mini-Appendix: Proof of (8.6).

To prove this claim we will need to distinguish two cases: Either $\sigma = 4$ or $\sigma > 4$. We will start with the case $\sigma > 4$ which is the easiest.

[124] See the introduction for a discussion of these notions.

[125] See [6] for a strict definition of this notion–see also the introduction of the present paper for a simplified description of this notion.

Proof of (8.6) when $\sigma > 4$: In this setting, we refer back to the grand conclusion. For each tensor field $C^{h,i_1\dots i_a,i_*}_g\nabla_{i_*}\phi_{u+1}$, $h \in H^{\S\S}$ we define

$$X^\sharp div_{i_1}\dots X^\sharp div_{i_a} C^{h,i_1\dots i_a,i_*}_g\nabla_{i_*}\phi_{u+1}$$

to stand for the sublinear combination in $Xdiv_{i_1}\dots Xdiv_{i_a}C^{h,i_1\dots i_a,i_*}_g\nabla_{i_*}\phi_{u+1}$ where each ∇_{i_v} is not allowed to hit *either of the two* factors $\nabla\Omega_x, \nabla\Omega_{x'}$ (which are contracting against each other).

We may then straightforwardly use the fact that the grand conclusion holds formally to derive an equation:

$$\begin{aligned}&\sum_{h\in H\bigcup H^{\S\S}} a_h X^\sharp div_{i_1}\dots X^\sharp div_{i_a} C^{h,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+\\&\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}) = 0,\end{aligned} \tag{8.13}$$

where $\sum_{j\in J}\dots$ above stands for a generic linear combination of complete contractions of length $\sigma+u+1$ with a weak character $Weak(\vec{\kappa}^+_{simp})$, with two factors $\nabla\Omega_x, \nabla\Omega_{x'}$ contracting against each other and which are *simply subsequent* to $\vec{\kappa}_{simp}$.

Now, we state a Lemma (which will be applied to other settings in the future), which fits perfectly with the equation above:

Lemma 8.1 *Consider a linear combination of tensor fields, $\sum_{\tau\in T} a_\tau C^{\tau,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_{p'},\phi_1,\dots,\phi_{u'})$, each with a given simple character $\overline{\kappa}_{simp}$, and each with $a \geq V$ (for some given V). We assume that this simple character falls under the inductive assumption of Proposition 1.1.*

We consider the tensor fields $C^{\tau,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_{p'},\phi_1,\dots,\phi_{u'})\nabla_i\Omega_x\nabla^i\phi_q$ which arise from the above by just multiplying by $\nabla_i\Omega_x\nabla^i\phi_q$. We assume an equation:

$$\begin{aligned}&\sum_{\tau\in T} a_\tau X^\sharp div_{i_1}\dots X^\sharp div_{i_a}[C^{\tau,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_{p'},\phi_1,\dots,\phi_{u'})\nabla_i\Omega_x\nabla^i\phi_q]+\\&\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_{p'},\phi_1,\dots,\phi_{u'})\nabla_i\Omega_x\nabla^i\phi_q = 0,\end{aligned} \tag{8.14}$$

where $X^\sharp div_i$ stands for the sublinear combination in $Xdiv_i$ where ∇_i is in addition not allowed to hit the expression $\nabla_i\Omega_x\nabla^i\phi_q$. $\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_{p'},\phi_1,\dots,\phi_{u'})$ stands for a generic linear combination of complete contractions with a weak character $Weak(\overline{\kappa}_{simp})$ and simply subsequent to $\overline{\kappa}_{simp}$. Furthermore, any terms of rank μ must have all μ free indices being non-special.

Our conclusion is then that we can write:

$$\sum_{\tau\in T} a_\tau Xdiv_{i_1}\dots Xdiv_{i_a}[C^{\tau,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_{p'},\phi_1,\dots,\phi_{u'})\nabla_i\Omega_x\nabla^i\phi_q] =$$
$$\sum_{\tau\in T'} a_\tau Xdiv_{i_1}\dots Xdiv_{i_a}C^{\tau,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_{p'},\phi_1,\dots,\phi_{u'}|\Omega_x,\phi_q)$$
$$+\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_{p'},\phi_1,\dots,\phi_{u'})\nabla_i\Omega_x\nabla^i\phi_q, \tag{8.15}$$

where the linear combination $\sum_{j\in J}\dots$ stands for a generic linear combination in the form described above. On the other hand, the linear combination $\sum_{\tau\in T'}$ stands for a generic linear combination of tensor fields where we have two factors $\nabla^{(A)}\Omega_x, \nabla^{(B)}\phi_q$ with $A=2, B=1$ or $A=1, B=2$ respectively, and in each case the term with one derivative is contracting against the other term (with two derivatives).

We claim that the above Lemma 8.1 (which we will prove momentarily), when applied to (8.13) implies our claim on the sublinear combination $\sum_{h\in H^{\S\S}}\dots$, in the case $\sigma>4$. This follows immediately once we set $\phi_q=\Omega_{x'}$, and once we observe that the tensor fields we obtain from (8.13) by *erasing* the expression $\nabla_i\Omega_x\nabla^i\Omega_{x'}$ have a simple character that falls under the inductive assumption of Proposition 1.1 (because we are increasing the weight).

Proof of Lemma 8.1:
The proof follows the usual inductive scheme:
We will assume that for some $A\ge V$, we can write:

$$\sum_{\tau\in T} a_\tau Xdiv_{i_1}\dots Xdiv_{i_a}[C^{\tau,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_{p'},\phi_1,\dots,\phi_{u'})\nabla_i\Omega_x\nabla^i\phi_q] =$$
$$\sum_{\tau\in T^A} a_\tau Xdiv_{i_1}\dots Xdiv_{i_a}[C^{\tau,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_{p'},\phi_1,\dots,\phi_{u'})\nabla_i\Omega_x\nabla^i\phi_q]+$$
$$\sum_{\tau\in T'} a_\tau Xdiv_{i_1}\dots Xdiv_{i_a}C^{\tau,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_{p'},\phi_1,\dots,\phi_{u'}|\Omega_x,\phi_q)$$
$$+\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_{p'},\phi_1,\dots,\phi_{u'})\nabla_i\Omega_x\nabla^i\phi_q, \tag{8.16}$$

where $\sum_{\tau\in T^A}$ on the RHS stands for a linear combination of tensor fields which are in the general form of the tensor fields in $\sum_{\tau\in T}$, only with rank $\ge A$. We can use the above to replace $\sum_{\tau\in T}\dots$ in our Lemma hypothesis by $\sum_{\tau\in T^A}\dots$. We will then show that we can write:

$$\sum_{\tau\in T^A} a_\tau Xdiv_{i_1}\dots Xdiv_{i_a}[C^{\tau,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_{p'},\phi_1,\dots,\phi_{u'})\nabla_i\Omega_x\nabla^i\phi_q]=$$
$$\sum_{\tau\in T^{A+1}} a_\tau Xdiv_{i_1}\dots Xdiv_{i_a}[C^{\tau,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_{p'},\phi_1,\dots,\phi_{u'})\nabla_i\Omega_x\nabla^i\phi_q]+$$
$$\sum_{\tau\in T'} a_\tau Xdiv_{i_1}\dots Xdiv_{i_a}C^{\tau,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_{p'},\phi_1,\dots,\phi_{u'}|\Omega_x,\phi_q)$$
$$+\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_{p'},\phi_1,\dots,\phi_{u'})\nabla_i\Omega_x\nabla^i\phi_q, \tag{8.17}$$

with the same notational conventions. Clearly, if we can show that (8.16) implies (8.17) then by iterating this step, we will derive our Lemma 8.1.

Proof that (8.16) implies (8.17): As explained, we may assume that $\sum_{\tau\in T}\dots=\sum_{\tau\in T^A}$. We denote by $T^A_*\subset T^A$ the index set of tensor fields with rank exactly A. Then, applying the eraser to the expression $\nabla_i\Omega_x\nabla^i\phi_q$, we derive an equation:

$$\begin{aligned}&\sum_{\tau\in T^A} a_\tau Xdiv_{i_1}\dots Xdiv_{i_a}[C^{\tau,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_{p'},\phi_1,\dots,\phi_{u'})]+\\&+\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_{p'},\phi_1,\dots,\phi_{u'})=0.\end{aligned} \tag{8.18}$$

Then, with certain exceptions,[126] we may apply Corollary 1 in [6] to the above, and derive that there is some linear combination of acceptable $(A+1)$-tensor fields with a u'-simple character $\vec{\kappa}'_{simp}$ (indexed in H below) so that:

$$\begin{aligned}&\sum_{\tau\in T^A_*} a_\tau C^{\tau,i_1\dots i_A}_g(\Omega_1,\dots,\Omega_{p'},\phi_1,\dots,\phi_{u'})\nabla_{i_1}\upsilon\dots\nabla_{i_A}\upsilon-\\&\sum_{\tau\in T^A_*} a_\tau Xdiv_{i_{A+1}}C^{\tau,i_1\dots i_{A+1}}_g(\Omega_1,\dots,\Omega_{p'},\phi_1,\dots,\phi_{u'})\nabla_{i_1}\upsilon\dots\nabla_{i_A}\upsilon=\\&\sum_{j\in J} a_j C^{j,i_1\dots i_A}_g(\Omega_1,\dots,\Omega_{p'},\phi_1,\dots,\phi_{u'})\nabla_{i_1}\upsilon\dots\nabla_{i_A}\upsilon\end{aligned} \tag{8.19}$$

(the tensor fields indexed in J are simply subsequent to $\vec{\kappa}'_{simp}$).[127]

[126] These exceptions are when there are tensor fields in T^A_* which are "forbidden tensor fields of Corollary 1 in [6] with rank $m\ge\mu+1$. (It follows that the forbidden tensor fields of rank μ cannot arise here, since all μ free indices of the tensor fields in $H^{\S\S}$ must have all their free indices being non-special).

[127] In the exceptional cases above, our claim (8.17) follows from from the "weak version" of Proposition 1.1 presented in [6], with $\Phi=\nabla_s\Omega_x\nabla^s\phi_q$.

Now, if we multiply the above by $\nabla_i\Omega_x\nabla^i\phi_q$ and make the $\nabla\upsilon$s into $Xdiv$'s (which *are* allowed to hit either of the factors $\nabla\Omega_x, \nabla\phi_q$), then we derive (8.17) and hence also Lemma 8.1. □

The proof of (8.6) in the case $\sigma = 4$:

In this case, we will prove (8.6) directly, immitating the ideas in [5]:

Proof of (8.6) in the case where $\sigma = 4$: We have that in this case, the tensor fields indexed in $H^{\S\S}$ will have two factors $\nabla\Omega_x, \nabla\Omega_{x'}$ (contracting against each other) *and two other factors*, which we denote by T_1, T_2.[128] We also recall that all tensor fields indexed in $H^{\S\S}$ have rank $\geq \mu$ (≥ 1), and if they do have rank μ they will also have a removable index, by construction. We distinguish the following cases regarding the *form* of the factors T_1: Either both these factors are of the form $\nabla^{(p)}\Omega_j$, or one (T_1, say) is in the form $\nabla^{(p)}\Omega_j$ and the other is a curvature term (either in the form $\nabla^{(m)}R_{ijkl}$ or $S_*\nabla^{(\nu)}R_{ijkl}$) or both T_1, T_2 are curvature factors (either in the form $\nabla^{(m)}R_{ijkl}$ or $S_*\nabla^{(\nu)}R_{ijkl}$). Label these cases A,B,C respectively.

In case A, we will assume with no loss of generality (up to re-labelling factors) that $T_1 = \nabla^{(c)}\Omega_1$, $T_2 = \nabla^{(c')}\Omega_2$ (and also $\Omega_x = \Omega_3, \Omega_{x'} = \Omega_4$). Then, by "manually" constructing divergences, we can show that:

$$\begin{aligned}&\sum_{h\in H^{\S\S}} a_h Xdiv_{i_1}\dots Xdiv_{i_a}[C^{h,i_1\dots i_a}_g(\Omega_1,\Omega_2,\phi_1,\dots,\phi_{u+1})\nabla_j\Omega_3\nabla^j\Omega_4] = \\ &\sum_{h\in H} a_h Xdiv_{i_1}\dots Xdiv_{i_a}[C^{h,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_4,\Omega\phi_1,\dots,\phi_{u+1})]+ \\ &(Const)_* Xdiv_{i_1}\dots Xdiv_{i_b}[C^{*,i_1\dots i_b}_g(\Omega_1,\Omega_2,\phi_1,\dots,\phi_{u+1})\nabla_j\Omega_3\nabla^j\Omega_4],\end{aligned} \tag{8.20}$$

where the tensor field $C^{*,i_1\dots i_b}_g(\Omega_1,\Omega_2,\phi_1,\dots,\phi_{u+1})\nabla_j\Omega_3\nabla^j\Omega_4$ is in the form:

$$pcontr(\nabla^{(A)}_{r_1\dots r_A}\Omega_1\otimes\nabla^{(B)}_{t_1\dots t_B}\Omega_2\otimes\nabla\phi_1\otimes\dots\otimes\nabla\phi_{u+1}),$$

with the following restrictions: All indices in both $\nabla^{(A)}\Omega_1, \nabla^{(B)}\Omega_2$ are either free or contracting against some $\nabla\phi_h$. Also, if we denote by β the number of factors $\nabla\phi_h$ that are contracting against $\nabla^{(B)}\Omega_2$ (notice β is encoded in $\vec{\kappa}_{simp}$, then $B = 2$ if $\beta \leq 2$, while $B = \beta$ if $\beta > 2$. The linear combination $\sum_{h\in H}\dots$ on the right hand side of the above stands for a *generic* linear combination of the form $\sum_{t\in T'}\dots$ allowed in the RHS of (8.6).

Then, using the above, we derive that $(Const) = 0$, and that concludes the proof of (8.6) in this case.

In case B, using the same technique of constructing "explicit" X-divergences, we derive an equation (8.20) only *without* the last term $(Const)_*\dots$. That immediately implies (8.6) in this case.

[128]In the case where both T_1, T_2 are generic terms in the form $\nabla^{(m)}R_{ijkl}$, the labelling T_1, T_2 is arbitrary; in all other cases, we will have a well-defined factor T_1 and a well-defined T_2.

Finally in case C, we distinguish subcases on the factors T_1, T_2: In subcase (i), both factors will be in the form $\nabla^{(m)}R_{ijkl}$. In subcase (ii), T_1 will be in the form $\nabla^{(m)}R_{ijkl}$ and T_2 will be in the form $S_*\nabla^{(\nu)}R_{ijkl}$. In subcase (iii), both T_1, T_2 will be in the form $S_*\nabla^{(\nu)}R_{ijkl}$.

Now, in subcase (i) we show (8.20) by the same argument, only now the tensor field $C^{*,i_1\dots i_b}_g(\Omega_1,\Omega_2,\phi_1,\dots,\phi_u)\nabla_j\Omega_3\nabla^j\Omega_4$ is in the form:

$$pcontr(\nabla^{(A)}_{r_1\dots r_A}R_{ijkl}\otimes\nabla^{(B)}_{t_1\dots t_B}R_{i'}{}^{jk}{}_{l'}\otimes\nabla\phi_1\otimes\dots\otimes\nabla\phi_{u+1}),$$

with the following restrictions: The indices ${}_{i,l,i',l'}$ are free. All derivative indices in both curvature factors are either free or contracting against some $\nabla\phi_h$. Also, if we denote by β the number of factors $\nabla\phi_h$ that are contracting against T_2 (notice β is encoded in $\vec{\kappa}_{simp}$, $B=2$ if $\beta\le 2$, while $B=\beta$ if $\beta>2$). We again derive that $(Const)=0$, which implies that (8.20) is our desired equation (8.6).

In subcase (ii) we use this technique to derive an equation:

$$\begin{aligned}&\sum_{h\in H^{\S\S}} a_h Xdiv_{i_1}\dots Xdiv_{i_a}[C^{h,i_1\dots i_a}_g(\Omega_1,\Omega_2,\phi_1,\dots,\phi_{u+1})\nabla_j\Omega_3\nabla^j\Omega_4]=\\&\sum_{h\in H} a_h Xdiv_{i_1}\dots Xdiv_{i_a}[C^{h,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_4,\phi_1,\dots,\phi_{u+1})]\\&+\sum_{j\in J}a_j[C^j_g(\Omega_1,\Omega_2,\phi_1,\dots,\phi_{u+1})\nabla_j\Omega_3\nabla^j\Omega_4],\end{aligned}\tag{8.21}$$

where the terms indexed in J are simply subsequent to $\vec{\kappa}_{simp}$.

Finally, in subcase (iii) we explicitly write:

$$\begin{aligned}&\sum_{h\in H^{\S\S}} a_h Xdiv_{i_1}\dots Xdiv_{i_a}[C^{h,i_1\dots i_a}_g(\Omega_1,\Omega_2,\phi_1,\dots,\phi_{u+1})\nabla_j\Omega_3\nabla^j\Omega_4]=\\&\sum_{h\in H} a_h Xdiv_{i_1}\dots Xdiv_{i_a}[C^{h,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_4,\Omega\phi_1,\dots,\phi_{u+1})]+\\&(Const)_* Xdiv_{i_1}\dots Xdiv_{i_b}[C^{*,i_1\dots i_b}_g(\Omega_1,\Omega_2,\phi_1,\dots,\phi_{u+1})\nabla_j\Omega_3\nabla^j\Omega_4]\\&+\sum_{j\imath J}a_j[C^j_g(\Omega_1,\Omega_2,\phi_1,\dots,\phi_{u+1})\nabla_j\Omega_3\nabla^j\Omega_4],\end{aligned}\tag{8.22}$$

where the tensor field $C^{*,i_1\dots i_b}_g(\Omega_1,\Omega_2,\phi_1,\dots,\phi_u)\nabla_j\Omega_3\nabla^j\Omega_4$ is in the form:

$$pcontr(S_*\nabla^{(A)}_{r_1\dots r_A}R_{ijkl}\otimes S_*\nabla^{(B)}_{t_1\dots t_B}R_{i'j'}{}^{k}{}_{l'}\otimes\nabla\phi_1\otimes\dots\otimes\nabla\phi_{u+1}),$$

with the following restrictions: The indices ${}_{l,l'}$ are free. All indices ${}_{r_1,\dots,r_A,j}$, ${}_{t_1,\dots,t_B,j'}$ are either free or contracting against some $\nabla\phi_h$. Also, if we denote by β the number of factors $\nabla\phi'_h$ that are contracting against T_2 (notice β is encoded in $\vec{\kappa}_{simp}$, then $B=1$ if $1\le\beta\le 2$, $B=0$ if $\beta=0$, while $B=\beta-1$ if $\beta>2$. We again derive that $(Const)=0$, which implies that (8.22) is our desired equation (8.6). $\Box$

8.4 Mini-Appendix: Proof of (7.23).

Firstly, we observe that for any tensor field $C_g^{l,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)$ with $a=\mu$ we will not have any of the free indices ${}_{i_i},\dots,{}_{i_\mu}$ belonging to any of the two factors $\nabla^{(2)}\phi_{u+1},\nabla^{(2)}\omega$ (this holds because we are in the setting of Lemma 1.3) hence no μ-tensor fields in (1.6) have special free indices in any factor $S_*\nabla^{(\nu)}R_{ijkl}$, while any free index in any of the factors $\nabla^{(2)}\phi_{u+1}$, $\nabla^{(2)}\omega_1$ would necessarily have arisen from a special free index in some factor S_*R_{ijkl} by (2.1).

Now, we apply the eraser to the factors $\nabla\phi_h$ that are contracting against $\nabla^{(2)}\phi_{u+1}$, $\nabla^{(2)}\omega$. We are left with factors $\nabla\phi_{u+1}$, $\nabla\omega$ and denote the tensor fields and complete contractions we are left with by $\overline{C}_g^{l,i_1\dots i_a}$, $\overline{C}_g^j$. Thus we obtain an equation:

$$\begin{aligned}&\sum_{l\in L^{\alpha,\beta}} a_l X_*div_{i_1}\dots X_*div_{i_a}\overline{C}_g^{l,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+\\&\sum_{j\in J^{\alpha,\beta,II}} a_j\overline{C}_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)=0\end{aligned}\tag{8.23}$$

(modulo complete contractions of length $\geq\sigma+u+1$).

We regard the factor $\nabla\omega$ as a factor $\nabla\phi_{u+2}$. We observe that the tensor fields $\overline{C}_g^{l,i_1\dots i_a}$ all have the same $(u-2)$-simple character (the one defined by the factors $\nabla\phi_h, h\leq u$), say $\overline{\kappa}_{simp}$ and each $\overline{C}_g^j$ is simply subsequent to that $(u-2)$-simple character. The tensor fields in the above either have rank either $a\geq\mu+1$ but may contain free indices in the factors $\nabla^{(2)}\phi_{u+1}$, $\nabla^{(2)}\omega$, *or* they have rank μ and in addition no free indices belong to the factors $\nabla^{(2)}\phi_{u+1},\nabla^{(2)}\omega$. We denote by $L^{\alpha,\beta,\sharp}$ the index set of tensor fields of rank exactly $\mu+1$ where both factors $\nabla^{(2)}\phi_{u+1}$, $\nabla^{(2)}\omega$ contain a free index (say the indices ${}_{i_1},{}_{i_2}$ wlog). We will prove that we can write:

$$\begin{aligned}&\sum_{l\in L^{\alpha,\beta,\sharp}} a_l Xdiv_{i_1}\dots Xdiv_{i_{\mu+1}}\overline{C}_g^{l,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)=\\&\sum_{l\in L'^{\alpha,\beta}} a_l Xdiv_{i_1}\dots Xdiv_{i_{\mu+1}}\overline{C}_g^{l,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+\\&\sum_{j\in J^{\alpha,\beta,II}} a_j\overline{C}_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)\end{aligned}\tag{8.24}$$

(where the tensor fields indexed in $L'^{\alpha,\beta}$ have all the features of the tensor fields indexed in $L^{\alpha,\beta}$ and in addition have the factor $\nabla^{(2)}\omega$ *not* containing a free index *and* with one index in the factor $\nabla^{(2)}\phi_{u+1}$ contracting against a non-special index.

Notice that if we can prove the above, then we are reduced to showing our claim under the additionnal assumption that $L^{\alpha,\beta,\sharp}=\emptyset$. Let us check how our

claim then follows under this assumtpion. We will then show (8.24) below.

Proof of our claim assuming (8.24): We break (8.23) into sublinear combinations with the same u-weak character[129] (suppose those sublinear combinations are indexed in the sets $L^{\alpha,\beta,v}$, $v \in V$); we derive an equation for each $v \in V$:

$$\begin{aligned}&\sum_{l \in L^{\alpha,\beta,v}} a_l X_* div_{i_1} \dots X_* div_{i_a} \overline{C}_g^{l,i_1\dots i_a}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)+\\ &\sum_{j \in J^{\alpha,\beta,II}} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega) = 0.\end{aligned} \tag{8.25}$$

Now, by virtue of the assumption $L^{\alpha,\beta,\sharp} = \emptyset$, we may assume that all the tensor fields in the above equation have rank $a \geq \mu$ and also have no free indices in the factors $\nabla\phi_{u+1}, \nabla\phi_{u+2}$. Then, applying Lemma 2.5 in [7][130] we derive (7.23). □

Proof of (8.24): We initially pick out the sublinear combination in (8.23) where both $\nabla\omega, \nabla\phi_{u+1}$ contract against the same factor T. Clearly this sublinear combination must vanish separately, and we will denote the new true equation that we thus obtain by New[(8.23)]. Thus, the sublinear combination of tensor fields indexed in $L^{\alpha,\beta}$ which contained at most one free index among the factors $\nabla\phi_{u+1}, \nabla\omega$ contributes a linear combination of iterated $Xdiv$'s of rank at least μ to New[(8.23)]. We denote the equation we have obtained by:

$$\begin{aligned}&\sum_{l \in L^{\alpha,\beta,\sharp}} a_l Doubdiv_{i_1 i_2} Xdiv_{i_3} \dots Xdiv_{i_{\mu+1}} \overline{C}_g^{l,i_1\dots i_{\mu+1}} +\\ &\sum_{f \in F} a_f Xdiv_{i_1} \dots Xdiv_{i_\mu} C_g^{f,i_1\dots i_\mu}(\nabla\phi_{u+1},\nabla\omega) + \sum_{j \in J} a_j C_g^j = 0\end{aligned} \tag{8.26}$$

($Doubdiv_{i_1 i_2}$ means that both derivatives $\nabla^{i_1}, \nabla^{i_2}$ are forced to hit the same factor). We then symmetrize the two factors $\nabla\phi_{u+1}, \nabla\omega$ and thus obtain a new true equation, which we denote by:

$$\begin{aligned}&\sum_{l \in L^{\alpha,\beta,\sharp}} a_l Doubdiv_{i_1 i_2} Sym[\overline{C}_g^{l,i_1\dots i_{\mu+1}}] Xdiv_{i_3} \dots Xdiv_{i_\mu} +\\ &\sum_{f \in F} a_f Xdiv_{i_1} \dots Xdiv_{i_\mu} Sym[C]_g^{f,i_1\dots i_\mu}(\nabla\phi_{u+1},\nabla\omega) + \sum_{j \in J} a_j Sym[C]_g^j = 0\end{aligned} \tag{8.27}$$

[129] The one defined by $\nabla\omega$ and $\nabla\phi_h$, $1 \leq h \leq u+1$, $h \neq \alpha, h \neq \beta$.

[130] A note to show why (8.25) *does not* fall under the "forbidden cases" of Lemma 2.5 in [7]: We observe that the tensor fields of minimum rank μ in (8.25) with both factors $\nabla\phi_{u+1}, \nabla\omega$ contracting against special indices can only arise from the μ-tensor fields in (1.6)–but those tensor fields will have no special free indices, thus (8.25) does not fall under a forbidden case of that Lemma.

(here $Sym[\dots]$ stands for the symmetrization over the two factors $\nabla\phi_{u+1}, \nabla\omega$).

We denote by $F^a \subset F$ the index set of tensor fields for which both the factors $\nabla\phi_{u+1}, \nabla\omega$ are contracting against internal indices in some factor $\nabla^{(m)}R_{ijkl}$. We denote by $F^b \subset F$ the index set of tensor fields for which one of the indices $\nabla\phi_{u+1}, \nabla\omega$ are contracting against a special index in some factor $S_*\nabla^{(\nu)}R_{ijkl}\nabla^i\tilde{\phi}_x$.[131] By replacing the two factors $\nabla_a\phi_{u+1}\nabla_b\omega$ by g_{ab} in the first case and the two factors $\nabla^i\tilde{\phi}_x\nabla^k\omega$ by g^{ik} in the second, and then using the operation $Ricto\Omega$ and iteratively applying Corollary 1 in [6],[132] we derive that we can write:

$$\sum_{f\in F^a\bigcup F^b} a_f Xdiv_{i_1}\dots Xdiv_{i_\mu} Sym[C]^{f,i_1\dots i_\mu}_g(\nabla\phi_{u+1},\nabla\omega) = \sum_{f\in F^{OK}} a_f Xdiv_{i_1}\dots Xdiv_{i_a} Sym[C]^{f,i_1\dots i_a}_g(\nabla\phi_{u+1},\nabla\omega) + \sum_{j\in J} a_j Sym[C]^j_g, \tag{8.28}$$

where the terms indexed in F^{OK} have all the properties of the terms indexed in F in (8.27) and in addition have at most one/none of the factors $\nabla\phi_{u+1}, \nabla\omega$ contracting against special indices in factors of the form $\nabla^{(m)}R_{ijkl}, S_*\nabla^{(\nu)}R_{ijkl}$. Therefore, we may assume that $F^a = F^b = \emptyset$ in (8.27).

Now, we refer to (8.27) and replace the expression $\nabla_a\phi_{u+1}\nabla_b\omega$ by g_{ab}; we denote the resulting equation by (8.27)'. We then apply Sub_ω to (8.27) (see the Appendix in [3]) (obtaining a new true equation which we denote by $D_g = 0$) and we then apply our inductive assumption of Lemma 4.10 in [6] to $D_g = 0$.[133] In order to describe the resulting equation, we just denote by $Cut[\overline{C}]^{l,i_1\dots i_{\mu+1}}_g$ the tensor field that arises from $\overline{C}^{l,i_1\dots i_{\mu+1}}_g$ by erasing the factor $\nabla_{i_1}\phi_{u+1}$ (along with the free index ${}_{i_1}$). We thus derive that there exists a linear combination of acceptable μ-tensor fields (indexed in K below), with a simple character $Cut(\vec{\kappa}_{simp})$ *and with the index* ${}_{i_{\mu+2}}$ *belonging to a real factor* so that:

[131] We will assume it is the factor $\nabla\omega$, wlog.

[132] A note to illustrate why the "forbidden cases" of Corollary 1 in [6] do not interfere with our argument: Observe that for the terms indexed in F^b there will be a removable index by construction, therefore the "forbidden cases" do not obstruct our iteration; the terms indexed in F^a with rank μ must necessarily have arisen from the tensor fields or rank μ in (1.6). Therefore they will have only non-special free indices, therefore at the first iteration, Corollary 1 in [6] can be applied. On the other hand, it is possible that at a subsequent step in the iteration we may obtain a "forbidden" tensor field of rank $> \mu$; in that case we use the "weak substitute" of Corollary 1 in [6], presented in the Appendix in [8].

[133] When we apply Lemma 4.10 in [6] we treat the $Xdiv_{i_1}[\dots\nabla_{i_1}\omega]$ as a linear combination of $(\mu-1)$-tensor fields–i.e. we "forget" the $Xdiv$ structure with respect to the factor $\nabla\omega$, thus the terms of minimum rank the factor $\nabla\omega$ contains a removable index, thus our assumption does not fall under a "forbidden case" of that Lemma.

$$\sum_{l\in L^{\alpha,\beta,\sharp}} a_l Xdiv_{i_2} Cut[\overline{C}]^{l,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\phi_{u+1},\omega)\nabla_{i_3}\upsilon\dots\nabla_{i_{\mu+1}}\upsilon$$
$$= \sum_{k\in K} a_k Xdiv_{i_{\mu+2}} C^{k,i_3\dots i_{\mu+2}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\omega)\nabla_{i_3}\upsilon\dots\nabla_{i_{\mu+1}}\upsilon. \tag{8.29}$$

Now, just multiplying the above by an expression $\nabla_s\phi_{u+1}\nabla^s\upsilon$ and then replacing the $\nabla\upsilon$s by $Xdiv$'s we derive (8.24). $\Box$

In section 9, we derive the other half of 1.3.

9 Proof of Lemma 1.3 in case B.

9.1 Introduction: A sketch of the strategy.

In order to derive Lemma 1.3 (which corresponds to case B of Lemma 3.5 in [6]) we will use all the tools that were developed in thi paper. Most importantly the "grand conclusion" but also the two separate equations that were added in order to derive it the"grand conclusion".

Main Strategy: For each μ-tensor field $C^{l,i_1\dots i_\mu}_g$, $l\in L^z, z\in Z'_{Max}$ in (1.6), we will *canonicaly* pick out a prescribed free index ${}_{i_1}$. We then consider the $(\mu-1)$-tensor fields $C^{l,i_1\dots i_\mu}_g\nabla_{i_1}\phi_{u+1}$, $l\in\bigcup_{z\in Z'_{Max}} L^z$.[134] We then prove (schematicaly) that there will exist a linear combination of $(\mu+1)$-tensor fields, $\sum_{h\in H} a_h C^{h,i_1\dots i_{\mu+1}}_g\nabla_{i_1}\phi_{u+1}$, each $C^{h,i_1\dots i_{\mu+1}}_g$ a partial contraction in the form (1.5), with the same u-simple character $\vec{\kappa}_{simp}$, such that:

$$\begin{aligned}&\sum_{l\in L^z} a_l Xdiv_{i_2}\dots Xdiv_{i_\mu} C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1} =\\ &\sum_{h\in H} a_h Xdiv_{i_2}\dots Xdiv_{i_{\mu+1}} C^{h,i_1\dots i_{\mu+1}}_g\nabla_{i_1}\phi_{u+1}\\ &+\sum_{j\in J} a_j C^{j,i_1}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1};\end{aligned} \tag{9.1}$$

here the terms indexed in J are "junk terms"; they have length $\sigma+u$ (like the tensor felds indexed in L_1 and H) and are in the general form (1.8). They are "junk terms" because one of the factors $\nabla\phi_h, 1\le h\le u$) which are supposed to contract against the index ${}_i$ in some factor $S_*\nabla^{(\nu)}R_{ijkl}$ for all the tensor fields indexed in L_μ now contracts against a derivative index of some factor

[134] These $(\mu-1)$-tensor fields arise from $C^{l,i_1\dots i_\mu}_g$ by just contracting the free index ${}_{i_1}$ against a new factor $\nabla\phi_{u+1}$.

$\nabla^{(m)} R_{ijkl}$.[135]

After we have derived an equation of the form (9.1), Lemma 1.3 follows by just applying the inductive claim of Proposition 1.1 to the above.

In order to derive (9.1), we will subdivide case B of Lemma 1.3 into subcases and treat them separately. In certain cases we must derive *systems* of equations combining the "grand conclusion" with other equations that we derived above. In certain very special subcases (such as when $\mu = 1$ in (1.6)), we will resort to ad hoc methods to derive (9.1).

We wish to stress again that when proving of Lemma 1.3 we are stll making all the inductive asumptions (on the parameters, $n, \sigma, \Phi, \sigma_1 + \sigma_2$) regarding the validity of Proposition 1.1 and also all of its consequences. Hence we are allowed to apply our inductive assumption of Proposition 1.1 or Lemmas 4.6, 4.8 etc from [6]

The subcases of Lemma 1.3: We distinguish subcases for Lemma 1.3 according to the maximal refined double character among the μ-tensor fields in (1.6). We refer the reader toto the introduction for a loose discussion of the notion of maximal refined double characters.[136] In particular, we recall that the maximal refined double character contains a decreasing list of numbers, $\vec{R\lambda}_{Max}$, which corresponds to the distribution of free indices among the different factors in the μ-tensor fields in $\bigcup_{z\in Z'_{Max}} L^z$. We have denoted $\vec{R\lambda}_{Max} = (M, B_1, \dots, B_\pi)$. The subcases are then as follows:

1. $M = 1$, $\pi > 0$.
2. $M \geq 2$ and $B_1 = \dots B_\pi = 1$, $\pi > 1$.
3. $M = \mu \geq 3$.
4. $M = \mu - 1 \geq 2$.
5. $M = \mu = 2$.
6. $M = \mu = 1$.

Technical remarks regarding the "grand conclusion": The "grand conclusion" is a new local equation, which is a consequence of (1.6); it applies in the setting of Lemma 1.3. It will be one of the main tools in deriving Lemma 1.3 in the present paper. For the reader's convenience, we recall a certaine conventions which we hav introduced:

[135] In the formal language introduced in [6], in this second scenario we would say that $C^{j,i_1}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)$ is "simply subequent" to the simple character $\vec{\kappa}_{simp}$.
[136] The proper definition appears in [6]

Recall conventions: Recall firstly that the "grand conclusion" is derived once we specify a particular factor/set of factors in $\vec{\kappa}_{simp}$.[137] This is called the "selected factor"/"selected set of factors". Given such a choice of factor/set of factors, we construct a new $(u+1)$-simple character $\vec{\kappa}^+_{simp}$ by contracting the chosen factor/one of the set of chosen factors against a new factor $\nabla\phi_{u+1}$; the new factor $\nabla\phi_{u+1}$ must *not* contract against a special index. Given such a choice of chosen factor/set of factors (and thus a new $(u+1)$-simple character), the grand conclusion will involve tensor fields in the form (1.5) with a $(u+1)$-simple character $\vec{\kappa}^+_{simp}$ and complete contractions in the form (1.8) with a weak $(u+1)$-character $Weak(\vec{\kappa}_{simp})$. When the selected factor is in the form $S_*\nabla^{(\nu)}R_{ijkl}$, the grand conclusion is the equation (8.3). When the selected factor is in the form $\nabla^{(m)}R_{ijkl}$ it is the equation (8.4), while when it is in the form $\nabla^{(A)}\Omega_h$ it is the equation (8.5).

9.2 A useful technical Lemma.

The next Lemma will allow us to assume wlog that all the tensor fileds indexed in H in the grand conclusion (i.e. all "contributors" there) are acceptable, and have the factor $\nabla\phi_{u+1}$ *not* contracting against a special index.

The "Technical Lemma" below has certain "forbidden cases", which we spell out here for reference purposes. A tensor field $C^{l,i_1\dots i_\mu}_g$ in (1.6) is "forbidden" (for the purposes of the next Lemma) if it has $\sigma_2 > 0$, each of the μ free indices belonging to a different factor, all factors $\nabla^{(\nu)}R_{ijkl}/\nabla^{(p)}\Omega_h$ must contract against none/at most one factor $\nabla\phi_y$, and *either* there are no removable free indices,[138] *or* there is exaclty one removable free index.[139]

Remark: There "forbidden cases" will only force us to give a special proof of Lemma 1.3 in the subcase $\mu = 1$, when the terms in (1.6) are "forbidden" as defined above. Those cases will be treated in the Mini-Appendix at the end of this paper.

Lemma 9.1 *Refer to the grand conclusion. We denote by $H_{Bad,1} \subset H$ the index set of tensor fields in H which have an unacceptable factor $\nabla\Omega_h$. We denote by $H_{Bad,2} \subset H$ the index set of tensor fields in H which have the factor $\nabla\phi_{u+1}$ contracting against a special index,*[140] *if $\sigma \geq 4$ (if $\sigma = 3$ we just set $H_{Bad,2} = \emptyset$).*

[137] In particular, "specifying one factor" means that we pick out a factor in (1.5) which is either in the form, $\nabla^{(B)}\Omega_x$, for some given x, or in the form $S_*\nabla^{(\nu)}R_{ijkl}\nabla^i\tilde{\phi}_h$ for some given h, or in the form $\nabla^{(m)}_{r_1\dots r_m}R_{ijkl}\nabla^{r_a}\phi_{h'}$ for some given h'. Specifying a "set of factors" means that we pick out the set of factors $\nabla^{(m)}R_{ijkl}$ in $\vec{\kappa}_{simp}$ which are not contracting against any factor $\nabla\phi_h$.

[138] (See definition 4.1 in [6]). In this setting, all factors must be in the form R_{ijkl}, S_*R_{ijkl} without derivatives, *or* in the form $\nabla^{(2)}\Omega_h$.

[139] One way to think of this is that for such a μ-tensor field one free index belongs to a factor $\nabla_{(free)}R_{\sharp\sharp\sharp\sharp}$ or $\nabla^{(3)}_{(free)\sharp\sharp}\Omega_h$, and *all* its other factors are in the form R_{ijkl}, S_*R_{ijkl} or $\nabla^{(2)}\Omega_h$.

[140] Therefore, these tensor fields will be acceptable by Definition 4.1.

Then, (unless the tensor fields of maximal refined double character in (1.6) are in one of the "forbidden forms" above) we claim that we can write:

$$\sum_{h\in H_{Bad,1}\bigcup H_{Bad,2}} a_h Xdiv_{i_1}\dots Xdiv_{i_a}C_g^{h,i_1\dots i_a,i_*}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}$$
$$=\sum_{h\in H_{OK}} a_h Xdiv_{i_1}\dots Xdiv_{i_a}C_g^{h,i_1\dots i_a,i_*}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+$$
$$\sum_{j\in J} a_j C_g^{j,i_*}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}. \tag{9.2}$$

The terms indexed in H_{OK} in the RHS stand for a generic linear combination of acceptable contributors[141] *with a $(u+1)$-simple character $\vec{\kappa}^+_{simp}$. The terms indexed in J are u-simply subsequent to $\vec{\kappa}_{simp}$.*

We observe that if we can show the above then we can assume wlog that all tensor fields in the grand conclusion are acceptable and have a $(u+1)$-simple character $\vec{\kappa}^+_{simp}$.

Proof of Lemma 9.1: We divide the index set $H_{Bad,1}$ into subsets $H^\alpha_{Bad,1}$, $H^\beta_{Bad,1}$ according to whether the factor $\nabla\Omega_h$ is contracting against a factor $\nabla\phi_{u+1}$ or not, respectively.

We firstly pick out the sublinear combination in the grand conclusion with a factor $\nabla\Omega_h$ contracting against the factor $\nabla\phi_{u+1}$. We thus derive a new equation:

$$\sum_{h\in H^\alpha_{Bad,1}} a_h X_*div_{i_1}\dots X_*div_{i_a}C_g^{h,i_1\dots i_a,i_*}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}$$
$$+\sum_{j\in J} a_j C_g^{j,i_*}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}=0. \tag{9.3}$$

Then applying Lemma 4.1 from [6][142] or 4.2 from [6][143] we derive that we can write:

[141] See definition 4.1.

[142] The fact that we have excluded the forbidden cases ensures that the terms of minimum rank in (9.3) does not fall under the "forbidden case" of that Lemma.

[143] The fact that we have excluded the forbidden cases ensures that the terms of minimum rank in (9.3) does not fall under the "forbidden case" of that Lemma.

$$\sum_{h\in H^{\alpha}_{Bad,1}} a_h Xdiv_{i_1}\dots Xdiv_{i_a} C^{h,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}$$
$$= \sum_{h\in H_{OK}} a_h Xdiv_{i_1}\dots Xdiv_{i_a} C^{h,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1} \tag{9.4}$$
$$+\sum_{j\in J} a_j C^{j,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}.$$

Thus, we are reduced to the case $H^{\alpha}_{Bad,1}=\emptyset$. Now, we pick out the sublinear combination in the grand conclusion with a factor $\nabla\Omega_h$ not contracting against $\nabla\phi_{u+1}$. We thus derive an equation:

$$\sum_{h\in H^{\beta}_{Bad,1}} a_h X_*div_{i_1}\dots X_*div_{i_a} C^{h,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}$$
$$+\sum_{j\in J} a_j C^{j,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}=0. \tag{9.5}$$

Now applying Corollary 4.6 in [6][144] (if $\sigma\ge 4$) or Lemma 4.7 in [6] (if $\sigma=3$)[145] to the above we derive that we can write:

$$\sum_{h\in H^{\beta}_{Bad,1}} a_h Xdiv_{i_1}\dots Xdiv_{i_a} C^{h,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}=$$
$$\sum_{h\in H_{OK}} a_h Xdiv_{i_1}\dots Xdiv_{i_a} C^{h,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}+$$
$$\sum_{j\in J} a_j C^{j,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}. \tag{9.6}$$

Thus, we may additionally assume that $H^{\beta}_{Bad,1}=\emptyset$. Finally, applying Lemma 4.10 in [6][146] to the above we derive that we can write:

$$\sum_{h\in H_{Bad,2}} a_h Xdiv_{i_1}\dots Xdiv_{i_a} C^{h,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}$$
$$= \sum_{h\in H_{OK}} a_h Xdiv_{i_1}\dots Xdiv_{i_a} C^{h,i_1\dots i_a,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1} \tag{9.7}$$
$$+\sum_{j\in J} a_j C^{j,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}.$$

[144] The fact that we have excluded the forbidden cases ensures that the terms of minimum rank in (9.3) does not fall under the "forbidden case" of that Lemma.

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[146] The fact that we have excluded the forbidden cases ensures that the terms of minimum rank in (9.3) does not fall under the "forbidden case" of that Lemma.

This concludes the proof of our Lemma. $\Box$

9.3 Proof of Lemma 1.3 in the subcase $M = 1, \pi > 0$:

In this case, it follows by the definition of the maximal refined double character that *all* μ-tensor fields $C_g^{l,i_1\dots i_\mu}$ in L_μ must have $M = 1, \pi = \mu - 1 > 0$.

In this setting, we claim:

$$\sum_{l\in L_\mu} a_l \sum_{k=1}^{\mu} Xdiv_{i_1}\dots X\hat{div}_{i_k}\dots Xdiv_{i_\mu} C_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_k}\phi_{u+1}$$
$$+\sum_{h\in H\bigcup H'} a_h Xdiv_{i_1}\dots Xdiv_{i_a} C_g^{h,i_1\dots i_a i_{a+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{a+1}}\phi_{u+1}$$
$$+\sum_{j\in J} a_j C_g^{j,i_1}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}, \tag{9.8}$$

where the tensor fields indexed in H have the property that they are acceptable with a u-simple character $\vec{\kappa}_{simp}$ and they satisfy $a \geq \mu$. The tensor fields indexed in H' have $a \geq \mu$, and are contributors, see Definition 4.1 above; in particular they have a u-simple character $\vec{\kappa}_{simp}$ but they also have one unacceptable factor $\nabla\Omega_h$ (with only one derivative). The complete contractions in J are u-simply subsequent to $\vec{\kappa}_{simp}$.

We will now show how Lemma 1.3 can be derived from (9.8) in this subcase.

Lemma 1.3 follows from (9.8) in this subcase: Firstly, we observe that by breaking (9.8) into sublinear combinations with the same weak $(u+1)$-character, we obtain a new set of true equations (since (9.8) holds formally and the weak character is invariant under the permutations that make the LHS of (9.8) vanish formally). So, for each $z \in Z'_{Max}$ we pick out the sublinear combination in (9.8) with a $(u+1)$-weak character $Weak(\vec{\kappa}^z_{ref-doub})$. We assume with no loss of generality (just by re-labelling free indices) that the sublinear combination of contractions in the first line of (9.8) with weak character $Weak(\vec{\kappa}^z_{ref-doub})$ are the summands $k = 1,\dots,V_z$; we also denote by H_z, H'_z, J_z the index sets of contractions with a weak character $Weak(\vec{\kappa}^z_{ref-doub})$. We denote $V_z = V$ for brevity and thus derive an equation:

$$\sum_{l\in L_\mu} a_l \sum_{k=1}^{V} Xdiv_{i_1}\dots X\hat{div}_{i_k}\dots Xdiv_{i_\mu} C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_k}\phi_{u+1}$$
$$+\sum_{h\in H_z\bigcup H'_z} a_h Xdiv_{i_1}\dots Xdiv_{i_a} C^{h,i_1\dots i_a i_{a+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{a+1}}\phi_{u+1}$$
$$+\sum_{j\in J_z} a_j C^{j,i_1}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}. \tag{9.9}$$

Now, our aim is to apply the inductive assumption of Corollary 1 in [6] to (9.9). In order to do this, we first apply Lemma 9.1 to (9.9) to derive a new equation where $H=\emptyset$ (thus all tensor fields are acceptable and have the same $(u+1)$-simple character).

Thus, we apply the inductive assumption of Corollary 1 in [6] to (9.9):[147] For our chosen $z\in Z'_{Max}$ we derive that there is a linear combination of $(\mu+1)$-tensor fields with a refined double character $\vec{\kappa}^z_{ref-doub}$ (indexed in P below) so that:

$$\sum_{l\in L^z_\mu} a_l C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon-$$
$$\sum_{p\in P} a_P Xdiv_{i_{\mu+1}} C^{p,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon=$$
$$\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1},\upsilon^{\mu-1}); \tag{9.10}$$

here the contractions indexed in J are simply subsequent to $\vec{\kappa}_{simp}$. Now, setting $\phi_{u+1}=\upsilon$, we derive our claim in this case. □

Proof of (9.8): This equation just follows by considering the equation $Im^{1,\beta}_{\phi_{u+1}}[L_g]=0$ (see (7.61) and then replacing $\nabla\omega$ by an $Xdiv$, by virtue of the last Lemma in the Appendix of [3]. □

9.4 Proof of Lemma 1.3 in the subcase $M\geq 2, B_1=1,\pi>1$:

This subcase follows by a very similar reasoning. We arbitrarily pick out some $z\in Z'_{Max}$ and we will show our claim for the tensor fields indexed in L^z. If we can do this then by induction we can derive Lemma 1.3 in this subcase. We

[147] The above equation falls under the inductive assumption of Corollary 1 in [6] because we have increased the value of Φ, while keeping all the other parameters fixed. Observe that the tensor fields of minimum rank in (9.9) will have only non-special free indices. Thus there is no danger of falling under a “forbidden case” of that Corollary.

recall that for each $l \in L^z, z \in Z'_{Max}$, $C^{l,i_1\dots i_\mu}_g$ has one factor with $M > 1$ free indices and all other $\pi > 1$ factors that contain free indices will each contain only one free index. Therefore, by the definition of *maximal* refined double character (see the beginning of this section) for any *non-maximal* $C^{l,i_1\dots i_\mu}$, any given factor will contain at most $M-1$ free indices. For notational convenience we assume wlog that for each $l \in L^z$, the indices ${}_{i_1},\dots,{}_{i_M}$ in $C^{l,i_1\dots i_\mu}_g$ belong to the same factor.

We will prove our claim in this case by considering the equation $Im^{1,\beta}_{\phi_{u+1}}[L_g] = 0$ (i.e. (7.61)), where we now set $\omega = \phi_{u+2}$. In order to analyze this equation and derive our claim, we will introduce some notation:

Notation: We denote by $\vec{\kappa}^{++}_z$ the refined $(u+2,\mu-2)$-double character that arises from $\vec{L^z}$ as follows: Consider all the refined double characters of the $(\mu-2)$-tensor fields $C^{l,i_1\dots i_\mu}_g\nabla_{i_\alpha}\phi_{u+1}\nabla_{i_\beta}\phi_{u+2}$, $\alpha,\beta > M$. Let $\vec{\kappa}^{++}_z$ be the *maximal* such $(u+2,\mu-2)$-refined double character (if there are many such refined double characters we pick out one arbitrarily). We will write $\vec{\kappa}^{++}$ instead of $\vec{\kappa}^{++}_z$ for brevity.

We assume with no loss of generality (and only for notational convenience) that $C^{l,i_1\dots i_\mu}_g\nabla_{i_{\mu-1}}\phi_{u+1}\nabla_{i_\mu}\phi_{u+2}$ has a maximal refined double character $\vec{\kappa}^{++}$. We denote by

$$\sum_{l\in L_{\vec{\kappa}^{++}}} a_l C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_\alpha}\phi_{u+1}\nabla_{i_\beta}\phi_{u+2}$$

the sublinear combination in

$$\sum_{\alpha,\beta>M}\sum_{l\in L^z} a_l C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_\alpha}\phi_{u+1}\nabla_{i_\beta}\phi_{u+2}$$

that consists of complete contractions with a refined double character $\vec{\kappa}^{++}$. By definition, it follows that there is a *nonzero, universal* combinatorial constant $(Const)$,[148] for which:

$$\begin{aligned}&\sum_{l\in L_{\vec{\kappa}^{++}}} a_l C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon = \\ &(Const)\sum_{l\in L^z} a_l\cdot C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon.\end{aligned} \tag{9.11}$$

Now, refer to the grand conclusion; we observe that for each $l \in L^z$, any $(\mu-2)$-tensor field $C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_\alpha}\phi_{u+1}\nabla_{i_\beta}\phi_{u+2}$ with either $\alpha \le M$ or $\beta \le M$, has a weak character that is different from $Weak(\vec{\kappa}^{++})$. In addition, we observe by the definition of refined double characters that for each $l \in L_\mu\setminus L^z$ all the $(\mu-2)$-tensor fields $C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_\alpha}\phi_{u+1}\nabla_{i_\beta}\phi_{u+2}$ with a weak character $Weak(\vec{\kappa}^{++})$ have a $(u+2)$-simple character $Simp(\vec{\kappa}^{++})$

[148] By "universal" we mean that it depends only on the refined double character $\vec{L}^z$.

and a refined double character that is either subsequent to, or equipolent to $\vec{\kappa}^{++}$. Therefore, with these conventions if we pick out the sublinear combination with weak character $Weak(\vec{\kappa}^{++})$ (recall that this sublinear combination must vanish separately) in the equation $Im^{1,\beta}_{\phi_{u+1}}[L_g] = 0$ we derive a new equation:

$$\sum_{l\in L_{\vec{\kappa}^{++}}} a_l Xdiv_{i_1}\dots X\hat{d}iv_{i_\alpha}\dots X\hat{d}iv_{i_\beta}\dots Xdiv_{i_\mu} C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$$
$$\nabla_{i_\alpha}\phi_{u+1}\nabla_{i_\beta}\phi_{u+2} + \sum_{l\in L'} a_l Xdiv_{i_1}\dots X\hat{d}iv_{i_\alpha}\dots X\hat{d}iv_{i_\beta}\dots Xdiv_{i_\mu}$$
$$C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_\alpha}\phi_{u+1}\nabla_{i_\beta}\phi_{u+2}+$$
$$\sum_{h\in H} a_h Xdiv_{i_1}\dots X\hat{d}iv_{i_\alpha}$$
$$\dots X\hat{d}iv_{i_\beta}\dots Xdiv_{i_{\mu+1}} C^{h,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_\alpha}\phi_{u+1}\nabla_{i_\beta}\phi_{u+2}+$$
$$\sum_{j\in J} a_j C^{j,i_* i_{**}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}\nabla_{i_{**}}.\phi_{u+2} = 0. \tag{9.12}$$

Here the tensor fields indexed in L' are all acceptable and have a $(u+2,\mu-2)$-refined double character that is either subsequent or equipolent to $\vec{\kappa}^{++}$ (refer to [6] for the strict definition of this notion, or to the introduction of this paper for a rough description.). The complete contractions indexed in J are simply subsequent to $\vec{\kappa}_{simp}$. The tensor fields indexed in H have have a u-simple character $\vec{\kappa}_{simp}$ and rank $\ge \mu-1$, but they may have one or two unacceptable factor(s) $\nabla\Omega_x$ (and furthermore any such factors must be contracting against $\nabla\phi_{u+1},\nabla\phi_{u+2}$), and possibly one or both of the factors $\nabla\phi_{u+1},\nabla\phi_{u+2}$ may be contracting against an internal index in some factor $\nabla^{(m)}R_{ijkl}$ or an index ${}_{k},{}_{l}$ in $S_*\nabla^{(\nu)}R_{ijkl}$.

Now, by repeating *exactly* the same argument (just formally replacing any expression $\nabla_i\Omega_h\nabla^i\Omega_{h'}$ by $\nabla_i\Omega_h\nabla^i\phi_{u+1}\nabla_j\Omega_{h'}\nabla^j\phi_{u+2}$) that showed that we may "get rid" of the sublinear combination $\sum_{h\in H^{\S\S}}\dots$ (modulo introducing correction terms in the form $\sum_{h\in H^\S}\dots+\sum_{j\in J}\dots$[149]) in the "grand conclusion",[150] we may also assume that all tensor fields indexed in H in (9.12) have at most one factor $\nabla\Omega_h$.

Then, applying Lemma 4.1 in [6] (or Lemma 4.2 there if $\sigma = 3$) we may assume wlog that all tensor fields indexed in H in (9.12) have no factors $\nabla\Omega_h$.[151] Under that additional assumption, we may apply the generalized version of Lemma 4.10 in [6] if necessary, and additionally assume that in (9.12) no tensor fields indexed in H have a factor $\nabla\phi_{u+1}$ or $\nabla\phi_{u+2}$ contracting against a special index. We are then in a position to apply Proposition 1.1 to (9.12); we derive that there exists a linear combination of acceptable $(\mu-1)$-tensor fields (indexed in P below) with a $(u+2)$-simple character $Simp(\vec{\kappa}^{++})$ so that:

[149] That argument, did not depend on the "forbidden cases".
[150] See the the subsection after the "grand conclusion".
[151] Since $\mu\ge 4$, by weight considerations there is no danger of "forbidden cases".

$$\sum_{l\in L_{\vec{\kappa}++}} a_l C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\alpha}\phi_{u+1}\dots\nabla_{i_\beta}\phi_{u+2}\dots\nabla_{i_\mu}\upsilon+$$
$$\sum_{p\in P} a_p Xdiv_{i_{\mu+1}}C^{h,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\alpha}\phi_{u+1}\dots\nabla_{i_\beta}\phi_{u+2}$$
$$\dots\nabla_{i_\mu}\upsilon+\sum_{j\in J} a_j C^{j,i_*i_{**}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\upsilon^{\mu-2})\nabla_{i_*}\phi_{u+1}\nabla_{i_{**}}\phi_{u+2}=0; \tag{9.13}$$

(here the complete contractions indexed in J on the RHS are simply subsequent to the u-simple character $\vec{\kappa}_{simp}$.

Setting $\phi_{u+1}=\phi_{u+2}=\upsilon$ in the above we derive case B of Lemma 1.3 in this subcase.

9.5 Proof of Lemma 1.3 in the subcases $M=\mu\geq 3$ and $M=\mu-1\geq 2$.

These are more challenging subcases. We will first consider the case where $M=\mu(\geq 3)$. Our claim in the case $M=\mu-1\geq 2$ will follow by a simplification of the same ideas and will be discussed at the end of this proof.

The case $M=\mu$ corresponds to the setting where the μ-tensor fields $C^{l,i_1\dots i_\mu}_g$ of maximal refined double character have *all* the free indices ${}_{i_1},\dots {}_{i_\mu}$ belonging to the same factor. Therefore, the sets $L^z\subset L_\mu, z\in Z_{Max}$ that index the tensor fields of maximal refined double character can be easily described in this setting: Let us list by $F_1,\dots F_c$ all the non-generic factors in $\vec{\kappa}_{simp}$.[152] Let us denote by L_{c+1} the set of all generic factors on $\vec{\kappa}_{simp}$. Then we may number the maximal refined double characters of the μ-tensor fields appearing in (1.6) as $\vec{L}^z, z\in\{1,\dots,c,c+1\}$: The refined double character $\vec{L}^z, z\leq c$ stands for the refined double character that arises from the simple character $\vec{\kappa}_{simp}$ by assigning μ free indices to the factor F_z (all these free indices must be non-special).[153] The refined double character $\vec{L}^{c+1}$ stands for the refined double character that arises from the simple character $\vec{\kappa}_{simp}$ by assigning μ free indices to one of the generic factors $\nabla^{(m)}R_{ijkl}$.

So, in this case we observe that the index set $L^z, z\in Z'_{Max}$ corresponds to *one* of the index sets $L^z, c\in\{1,\dots,c+1\}$ introduced above. So we prove Lemma 1.3 for any chosen index set $L^z, z\in\{1,\dots,c+1\}$. So from now on z is a fixed number from the set $\{1,\dots,c+1\}$.

Now, we denote by T_z the factor in each $C^{l,i_1\dots i_\mu}_g$, $l\in L^z$ that contains the M free indices (recall that we have called this factor the *critical factor*). We

[152] Recall the "generic" factors in $\vec{\kappa}_{simp}$ are those factors in the form $\nabla^{(m)}R_{ijkl}$ that are *not* contracting against any factors $\nabla\phi_h$.

[153] Recall that free indices are called "special" if they are internal indices in some factor $\nabla^{(m)}R_{ijkl}$ or indices ${}_k,{}_l$ in some factor $S_*\nabla^{(\nu)}R_{ijkl}$.

recall that $\vec{L}^z$ stands for the (u,μ)-refined double character of the tensor fields $C^{l,i_1\dots i_\mu}_g$, $l\in L^z$.

We introduce some notation that will be useful for our proof:

Definition 9.1 *Consider the refined double characters that is formally constructed out of $\vec{L}^z$ by erasing one of the M free indices from the critical factor and then adding a free derivative index on some other factor $S_*\nabla^{(\nu)}R_{ijkl}$, or $\nabla^{(m)}R_{ijkl}$ or $\nabla^{(p)}\Omega_x$. We call those the linked refined double characters. We denote the list of those linked refined double characters by $\{\vec{\kappa}_1,\dots,\vec{\kappa}_\gamma\}$.*[154]

We then define $L_{\vec{\kappa}_1},\dots,L_{\vec{\kappa}_\gamma}\subset L_\mu$ to be the index sets of the tensor fields $C^{l,i_1\dots i_\mu}$ in (1.6) with a refined double character $\vec{\kappa}_1,\dots,\vec{\kappa}_\gamma$ respectively. For notational convenience, we assume that for each $C^{l,i_1\dots i_\mu}_g$ with a linked double character, the indices ${}_{i_1},\dots,{}_{i_{\mu-1}}$ belong to the critical factor T_z (and thus the free index ${}_{i_\mu}$ belongs to F_h which we will call the second critical factor).

Note: Let us consider any $C^{l,i_1\dots i_\mu}_g$ with a refined double character $\vec{\kappa}_h$, $1\le h\le\gamma$. Then, by the hypothesis $L^+_\mu=\emptyset$ of our Lemma 1.3, we can assume without loss of generality that the free index ${}_{i_\mu}$ will be a derivative index.

We now define the second linked refined double characters of each $\vec{L}^z$:

Definition 9.2 *Consider the refined double characters that are formally constructed out of $\vec{L}^z$ by erasing two free indices from the critical factor T_z and adding two free indices onto some other factor $\nabla^{(m)}R_{ijkl}$, $S_*\nabla^{(\nu)}R_{ijkl}$ (necessarily non-simple), $\nabla^{(p)}\Omega_h$. These (formally constructed) refined double characters will be the second linked refined double characters.*

Observe that there is an obvious correspondence between the linked and the second linked refined double characters of $\vec{L}^z$ based on the non-critical factor that we hit with one or two free derivative indices. Therefore, we denote the second linked refined double characters by $\vec{\kappa}'_1,\dots,\vec{\kappa}'_\gamma$, so that each $\vec{\kappa}'_h$ corresponds to $\vec{\kappa}_h$.

Technical remark concerning Definitions 9.1, 9.2: If either F_h or T_z are not generic factors $\nabla^{(m)}R_{ijkl}$, we will then have that $\vec{\kappa}_h\ne\vec{\kappa}'_h$ for every $h=1,\dots,\gamma$. If both F_z and T_z are generic factors $\nabla^{(m)}R_{ijkl}$ and $M\ge 4$ this is still true. In the case where both F_h,T_z are generic factors $\nabla^{(m)}R_{ijkl}$ not contracting against $\nabla\phi$'s and $M=3$ we have that $\vec{\kappa}_h=\vec{\kappa}'_h$. Furthermore, if $M=4$ and both F_h,T_z are generic we see that in $\vec{\kappa}'_h$ there is no well-defined defined critical factor. In that case, abusing language, when we refer to the critical factor we will in fact be counting $C^{l,i_1i_2i_3i_4}_g$ twice: Once with the the factor to which ${}_{i_1},{}_{i_2}$ belong being the critical factor and once with the factor to which ${}_{i_3},{}_{i_4}$ belong being the critical factor. Moreover, if $M=3$ and F_h,T_z are generic then $\vec{\kappa}_h=\vec{\kappa}'_h$. Therefore, when we speak of the complete contractions indexed in $L_{\vec{\kappa}_h}$ and $L_{\vec{\kappa}'_h}$ we are counting the same tensor fields twice. This, however, will not affect our

[154]Slightly abusing language we will say that $\vec{\kappa}_h$ arises from $\vec{L}^z$ by hitting the factor F_h by a derivative ∇_{i_*}.

conclusions further down.

Now, some more notation: We denote by $\vec{\kappa}^+_z$ the refined double character of the $(\mu-1)$-tensor fields $C^{l,i_1\dots i_\mu}_g\nabla_{i_1}\phi_{u+1}$, $l\in L^z$. For each $l\in L_{\vec{\kappa}_h}$, $h=1,\dots,\gamma$ (which is linked to $\vec{L^z}$), we denote by $\dot{C}^{l,i_1i_2\dots\hat{i}_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ the $(\mu-1)$-tensor field that arises from $C^{l,i_1\dots i_\mu}_g$ by erasing the index ${}_{i_\mu}$. We also denote by $\dot{C}^{l,i_1i_2\dots\hat{i}_\mu|i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ the $(\mu-1)$-tensor field that arises from $\dot{C}^{l,i_1i_2\dots\hat{i}_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ by adding a derivative index ∇_{i_*} onto the critical factor with $M-1$ free indices.

Now, we apply the grand conclusion to $L_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$, making T_z the *selected factor*. We derive an equation:

$$\begin{aligned}
&Q_z\cdot\sum_{l\in L^z}a_lXdiv_{i_2}\dots Xdiv_{i_\mu}C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\
&\sum_{y=1}^{\gamma}\overline{2}_y\sum_{l\in L_{\vec{\kappa}_y}}a_lXdiv_{i_1}\dots Xdiv_{i_{\mu-1}}[\dot{C}^{l,i_1i_2\dots\hat{i}_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}]\\
&+\sum_{l\in L'}a_lXdiv_{i_1}\dots Xdiv_{i_{\mu-1}}C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_\mu}\phi_{u+1}+\\
&\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_\mu}C^{h,i_1\dots i_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\
&\sum_{f\in F}a_fC^{f,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}=0,
\end{aligned}\tag{9.14}$$

modulo complete contractions of length $\geq\sigma+u+2$. Here Q_z stands for the first coefficient in the grand conclusion (it depends on the form of the factor T_z) and $|I_1|=\mu$. Also recall that $\overline{2}_y$ stands for 2 if F_y is of the form $\nabla^{(m)}R_{ijkl}$ or $S_*\nabla^{(\nu)}R_{ijkl}$ and 1 if F_y is of the form $\nabla^{(p)}\Omega_h$. The tensor fields indexed in L' are generic, acceptable $(\mu-1)$-tensor fields with a simple character $Simp(\vec{\kappa}^+_z)$ but a refined double character that is either doubly subsequent or equipolent to $\vec{\kappa}^+_z$. The tensor fields indexed in H are contributors (see Definition 4.1).

Now, by applying Lemma 9.1 to the above equation, we may assume with no loss of generality that all the tensor fields indexed in H are acceptable and have a $(u+1)$-simple character $\vec{\kappa}^+_z$.

Then, applying Corollary 1 in [6][155] to the above and picking out the sublinear combination of terms where all $\nabla\upsilon$'s are contracting against the factor T_z, we derive that there is a linear combination of acceptable μ-tensor fields (indexed in P below) with $(u+1,\mu-1)$-refined double character $\vec{\kappa}^+_z$ so that:

[155] There are no special free indices in the tensor fields of minimum rank, hence there is no danger of falling under "forbidden cases".

$$
\begin{aligned}
&Q_z \cdot \sum_{l \in L^z} a_l C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon+\\
&\sum_{y=1}^{\gamma}\overline{2}_y \sum_{l\in L_{\vec{\kappa}_y}} a_l[\dot{C}^{l,i_1i_2\dots\hat{i}_\mu,i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}]\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon\\
&-\sum_{p\in P} a_p Xdiv_{i_{\mu+1}} C^{p,i_1\dots i_\mu,i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon\\
&=\sum_{j\in J} a_j C^{j,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon,
\end{aligned}
\tag{9.15}
$$

where the tensor fields indexed in J are u-subsequent to $\vec{\kappa}_{simp}$.

Since $\mu \geq 3$ we may assume with no loss of generality that $\nabla\phi_{u+1}$ is contracting against a derivative index. By applying $Erase_{\phi_{u+1}}$ we obtain an equation:

$$
\begin{aligned}
&Q_z \cdot \sum_{l \in L^z} a_l C^{l,i_2\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon+\\
&\sum_{y=1}^{\gamma}\overline{2}_y \sum_{l\in L_{\vec{\kappa}_y}} a_l[\dot{C}^{l,i_1i_2\dots\hat{i}_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon\\
&-\sum_{p\in P} a_p Xdiv_{i_{\mu+1}} C^{p,i_2\dots i_\mu,i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon=\\
&\sum_{j\in J} a_j C^{j,i_2\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon,
\end{aligned}
\tag{9.16}
$$

where the tensor fields indexed in J are u-subsequent to $\vec{\kappa}_{simp}$.

Remark 1: Thus the above equation involves the μ-tensor fields indexed in L^z, which our Lemma 1.3 deals with in this subcase, but it *also* involves the μ-tensor fields in the second line. We can therefore *not* derive our Lemma 1.3 in this subcase from the above equation alone. We seek to derive two more equations in order to obtain a closed system of three equations in three different sublinear combinations. In order to formulate and derive our next two equations we will need to introduce some more notation:

Notation: For each $h, 1 \leq h \leq c+1$ we denote by $\dot{C}^{l,\hat{i}_1 i_2\dots i_\mu,i_*\to F_h}_g$ the tensor field that formally arises from $C^{l,i_1\dots i_\mu}_g$, $l \in L^z$ by erasing a derivative index $_{i_\mu}$ and hitting the (one of the) factor(s) F_h by a derivative ∇_{i_*} (and adding over all tensor fields we thus obtain if $h = c+1$). Also, for each $l \in L_{\vec{\kappa}'_v}$, $v = 1\dots,\gamma$, we denote by $C^{l,i_1\dots i_{\mu-1}\hat{i}_\mu|i_*}_g$ the tensor field that arises from $C^{l,i_1\dots i_\mu}_g$ by erasing

the free index ${}_{i_\mu}$,[156] (we may always assume it is a derivative index) and adding a derivative index ∇_{i_*} onto the critical factor T_z. So these tensor fields have $M-1$ free indices in T_z.

Now, we apply the grand conclusion to L_g making F_h the selected factor. In order to describe the equation we then obtain, we introduce some notation: Consider $(\mu-1)$-tensor fields with a factor $\nabla\phi_{u+1}$ contracting against a non-special index in F_h,[157] while all the other $\mu-1$ free indices belong to the factor T_z. We denote by $\vec{\kappa}^+_h$ the refined $(u+1,\mu-1)$-double character that corresponds to these tensor fields. In this setting we will denote by $\sum_{l\in L'} a_l C^{l,i_1\dots i_\mu}_g \nabla_{i_\mu}\phi_{u+1}$ a generic linear combination of tensor fields with a $(u+1)$-simple character $Simp(\vec{\kappa}^+_h)$ but a $(u+1,\mu-1)$-refined double character that is either doubly subsequent or equipolent to $\vec{\kappa}^+_h$. We derive:

$$
\begin{aligned}
&(\overline{2}_z M-\binom{M}{2})\sum_{l\in L^z} a_l Xdiv_{i_2}\dots Xdiv_{i_\mu}\dot{C}^{l,\hat{i}_1 i_2\dots i_\mu,i_*\to F_h}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\\
&\nabla_{i_*}\phi_{u+1}+Q_h\cdot\sum_{l\in L_{\vec{\kappa}_h}} a_l Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_\mu}\phi_{u+1}\\
&+\sum_{l\in L_{\vec{\kappa}'_h}} a_l Xdiv_{i_1}\dots Xdiv_{i_{\mu-2}} Xdiv_{i_*} C^{l,i_1\dots i_{\mu-1}\hat{i}_\mu|i_*}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\\
&\nabla_{i_{\mu-1}}\phi_{u+1}+\\
&\sum_{l\in L'} a_l Xdiv_{i_1}\dots Xdiv_{i_{\mu-1}} C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_\mu}\phi_{u+1}\\
&+\sum_{h\in H} a_h Xdiv_{i_1}\dots Xdiv_{i_a} C^{h,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})=\\
&\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),
\end{aligned}
\tag{9.17}
$$

where Q_h stands for the coefficient in the grand conclusion with selected factor F_h and $|I_1|=1$. Recall $\overline{2}_z$ stands for 2 if the factor T_z is of the form $\nabla^{(m)}R_{ijkl}$ or $S_*\nabla^{(\nu)}R_{ijkl}$. If T_z is of the form $\nabla^{(p)}\Omega_j$ then it stands for 1 if T_z is not contracting against a factor $\nabla\phi_h$ and 0 otherwise. The complete contractions in J are u-simply subsequent to $\vec{\kappa}_{simp}$.

Remark 2: Now, we observe that the equation above involves the sublinear combinations indexed in L^z, $L_{\vec{\kappa}_h}$, but it also involves the sublinear combination indexed in $L_{\vec{\kappa}'_h}$. Thus, we seek to derive a third equation in order to obtain a closed system.

[156] Recall that we are assuming ${}_{i_{\mu-1}}, {}_{i_\mu}$ belong to the second critical factor.

[157] Recall that a special index is an internal index in some factor $\nabla^{(m)}R_{ijkl}$ or an index ${}_{k},{}_{l}$ in some factor $S_*\nabla^{(\nu)}R_{ijkl}$.

Therefore, we invoke the equation $Im^{1,\beta}_{\phi_{u+1}}[L_g] = 0$. i.e. equation (7.61). We then define an operation $SimpOp$ that acts on the terms in $Im^{1,\beta}_{\phi_{u+1}}[L_g]$ by replacing the factor $\nabla\omega$ by an $Xdiv$. It follows that $SimpOp\{Im^{1,\beta}_{\phi_{u+1}}[L_g]\} = 0$ (by virtue of the last Lemma in the Appendix of [3]).

We will again pick any $h \le c+1$ and focus on the terms in $Im^{1,\beta}_{\phi_{u+1}}[L_g] = 0$ with the factor $\nabla\phi_{u+1}$ contracting against the factor F_h, and $\nabla\omega$ contracting against T_z. (So we again focus on the $(u+1, \mu-1)$-refined double character $\vec{\kappa}^+_h$ as before). Picking out this sublinear combination (which vanishes separately), and acting on it by $SimpOp[\dots]$, we derive:

$$\binom{M}{2} \sum_{l \in L^z} a_l Xdiv_{i_2} \dots Xdiv_{i_\mu} \dot{C}^{l,\hat{i}_1 i_2 \dots i_\mu, i_*}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)\nabla_{i_*}\phi_{u+1} +$$
$$(M-1) \sum_{l \in L_{\vec{\kappa}_h}} a_l Xdiv_{i_1} \dots Xdiv_{i_{\mu-1}} C^{l,i_1 \dots i_\mu}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)\nabla_{i_\mu}\phi_{u+1} +$$
$$\sum_{l \in L_{\vec{\kappa}'_h}} a_l Xdiv_{i_1} \dots Xdiv_{i_{\mu-1}} X\hat{div}_{i_\mu} Xdiv_{i_*} C^{l,i_1 \dots i_{\mu-1}|i_*}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)$$
$$\nabla_{i_{\mu-1}}\phi_{u+1} + \sum_{l \in L'} a_l Xdiv_{i_1} \dots Xdiv_{i_{\mu-1}} C^{l,i_1 \dots i_\mu}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)\nabla_{i_\mu}\phi_{u+1} +$$
$$\sum_{h \in H} a_h Xdiv_{i_1} \dots Xdiv_{i_a} C^{h,i_1 \dots i_a}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_{u+1}) =$$
$$\sum_{j \in J} a_j C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_{u+1}); \tag{9.18}$$

(the sublinear combinations in L', H, J stand for *generic* linear combinations as in (9.17)).

Now, we may first apply Lemma 9.1 (to ensure all tensor fields in H are acceptable and have the same $(u+1)$-simple character $\vec{\kappa}^+_h$), and then Corollary 1 in [6][158] to the two equations (9.17), (9.18), and pick out the sublinear combinations where there are $(\mu-1)$ factors $\nabla\upsilon$ contracting against T_z and $\nabla\phi_{u+1}$ is contracting against F_h. These sublinear combinations will vanish separately, and we thus derive two new equations:

[158] There will be $\mu - 1 \ge 2$ non-special free indices among the tensor fields of minimum rank, hence no danger of falling under a "forbidden case".

$$(\overline{2}_z M - \binom{M}{2}) \sum_{l\in L^z} a_l \dot{C}_g^{l,\hat{i}_1 i_2\dots i_\mu, i_*\to F_h}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}\nabla_{i_2}\upsilon\dots$$
$$\nabla_{i_\mu}\upsilon + Q_h\cdot \sum_{l\in L_{\vec{\kappa}_h}} a_l C_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_\mu}\phi_{u+1}\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon+$$
$$\sum_{l\in L_{\vec{\kappa}'_h}} a_l C_g^{l,i_1\dots i_{\mu-1}\hat{i}_\mu|i_*}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{\mu-1}}\phi_{u+1}\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-2}}\upsilon\nabla_{i_*}\upsilon+$$
$$\sum_{h\in H_1} a_h Xdiv_{i_{\mu+1}} C_g^{h,i_1\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon =$$
$$\sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1},\upsilon^{\mu-1}), \tag{9.19}$$

$$\binom{M}{2} \sum_{l\in L^z} a_l \dot{C}_g^{l,\hat{i}_1 i_2\dots i_\mu, i_*}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon+$$
$$(M-1)\sum_{l\in L_{\vec{\kappa}_h}} a_l C_g^{l,i_1\dots i_\mu}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_\mu}\phi_{u+1}\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon+$$
$$\sum_{l\in L_{\vec{\kappa}'_h}} a_l C_g^{l,i_1\dots i_{\mu-1}|i_*}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_{\mu-1}}\phi_{u+1}\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon+$$
$$\sum_{h\in H_2} a_h Xdiv_{i_{\mu+1}} C_g^{h,i_1\dots i_{\mu+1}}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon =$$
$$\sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1},\upsilon^{\mu-1}). \tag{9.20}$$

In the above two equations, all the tensor fields in the first three lines are acceptable, and have a $(u+1,\mu-1)$-refined double character $\vec{\kappa}^+_h$. The tensor fields indexed in H_1 are acceptable contributors with a $(u+1)$-simple character $\vec{\kappa}^+_h$ (see Definition 4.1). The complete contractions in J are in both cases simply subsequent to $Simp(\vec{\kappa}^+_h)$, and hence also to $\vec{\kappa}_{simp}$.

Subtracting (9.20) from (9.19) we derive a new equation:

$$(\overline{2}_z M - 2\binom{M}{2})\sum_{l\in L^z} a_l \dot{C}^{l,\hat{i}_1 i_2\dots i_\mu, i_*\to F_h}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\phi_{u+1}\nabla_{i_2}\upsilon\dots$$
$$\nabla_{i_\mu}\upsilon + (Q_h-(M-1))\cdot\sum_{l\in L_{\vec{\kappa}_h}} a_l C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_\mu}\phi_{u+1}\nabla_{i_1}\upsilon\dots$$
$$\nabla_{i_{\mu-1}}\upsilon + \sum_{h\in H'} a_h Xdiv_{i_{\mu+1}} C^{h,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon$$
$$=\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1},\upsilon^{\mu-1}), \tag{9.21}$$

where the tensor fields and complete contractions indexed in H', J have the same general properties as the ones indexed in H_1, J in (9.19).

We act on (9.21) by the operation $Erase_{\phi_{u+1}}$ (this operation is formally well-defined and produces acceptable tensor fields) and divide by $Q_h-(M-1)$. We thus derive an equation:

$$\begin{aligned}
&\sum_{l\in L_{\vec{\kappa}_h}} a_l C^{l,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon = \\
&\frac{(2\binom{M}{2}-2_z M)}{Q_h-(M-1)}\sum_{l\in L^z} a_l \dot{C}^{l,\hat{i}_1 i_2\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon + \\
&\sum_{p\in P} a_p Xdiv_{i_\mu} C^{p,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon + \\
&\sum_{j\in J} a_j C^{j,i_1\dots i_{\mu-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon.
\end{aligned} \tag{9.22}$$

Notice the coefficient $\frac{(2\binom{M}{2}-2_z M)}{Q_h-(M-1)}$ is *non-positive* (because $M\ge 3$ and $Q_h < M-1$ by inspection). We denote this coefficient by c_h.

Now, replacing the above into (9.16), and since $(Q_z+\sum_{h=1}^{\gamma} 2_h\cdot c_h)<0$, we obtain:

$$\begin{aligned}
&\sum_{l\in L^z} a_l C^{l,i_2\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon \\
&-\sum_{p\in P} a_p Xdiv_{i_{\mu+1}} C^{p,i_2\dots i_\mu,i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon = \\
&\sum_{j\in J} a_j C^{j,i_2\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_2}\upsilon\dots\nabla_{i_\mu}\upsilon,
\end{aligned} \tag{9.23}$$

modulo complete contractions of length $\ge \sigma+u$.

Now, define an operation Op that acts on the above by adding a derivative ∇_i onto the factor T_z which is contracting against the $\mu - 1$ factors $\nabla \upsilon$, snd contracting ∇_i against a factor $\nabla \upsilon$. Since (9.23) holds formally, Op produces a true equation. Applying Lemma 9.1 to this equation, we derive our desired conclusion for the case $M = \mu \geq 3$.

We now prove case B of Lemma 1.3 in the case, $M = \mu - 1 \geq 2$. Slightly abusing notation, we carry over the notation from the previous discussion, only now $L^z = \emptyset$. In this convention, the μ-tensor fields in (1.6) of maximal refined double character will correspond to the index sets $L_{\vec{\kappa}'_h}$, $h = 1, \dots, \gamma$.

We can again make use of equations (9.17) and (9.18), where we now have that $L^z = \emptyset$. We then subtract these two equations, obtaining a new true equation. Since $Q_h - (M-1) < 0$ as before, we may divide by that constant and set $\phi_{u+1} = \upsilon$. That gives us our desired conclusion. $\Box$

9.6 Proof of Lemma 1.3 in the subcase $M = \mu = 2$.

As before, the two main ingredients from the earlier discussion that we will be using here are the grand conclusion, and also the equation (7.61), $Im^{1,\beta}_{\phi_{u+1}}[L_g] = 0$.

This case presents certain particular difficulties. We first divide $L_\mu = L_2$ into two subsets: We say $l \in L^{1,1}$ if the two free indices ${}_{i_1}, {}_{i_2}$ belong to different factors. We say $l \in L^{2,0}$ if the two free indices ${}_{i_1}, {}_{i_2}$ belong to the same factor. We recall our Lemma hypothesis $L^*_\mu = \emptyset$ which ensures that for each $l \in L^{2,0}$ we cannot have ${}_{i_1}, {}_{i_2}$ belonging to a factor $\nabla^{(2)}\Omega_h$.

Firstly let us understand what the refined double characters associated to $\vec{\kappa}_{simp}$ are. This is easy to do, in this case: We denote by $F_1, \dots, F_\tau$ the list of factors in $\vec{\kappa}_{simp}$ which are contracting against a factor $\nabla\phi_h$ or are of the form $\nabla^{(p)}\Omega_y$. The rest of the factors F_l in $\vec{\kappa}_{simp}$ that do not belong to this list will be generic factors of the form $\nabla^{(m)}R_{ijkl}$.[159] Now, for each $h \leq \tau$ we may unambiguously speak of *the* factor F_h.

For each $h \leq \tau$, we denote by $L^{2,0|h} \subset L^{2,0}$ the index set of the 2-tensor fields $C^{l,i_1 i_2}_g$ with the two free indices ${}_{i_1}, {}_{i_2}$ belonging to the factor F_h. On the other hand, we index in $L^{2,0|\tau+1} \subset L^{2,0}$ all the 2-tensor fields which have the two indices ${}_{i_1}, {}_{i_2}$ belonging to a generic factor $\nabla^{(m)}R_{ijkl}$. We observe that:

$$\sum_{l \in L^{2,0}} a_l C^{l,i_1 i_2}_g = \sum_{h=1}^{\tau+1} \sum_{l \in L^{2,0|h}} a_l C^{l,i_1 i_2}_g.$$

Notice that (in this subcase) the 2-tensor fields of *maximal* refined double character (in (1.6)) are the ones indexed in $L^{2,0}$. Observe that the index set $\bigcup_{z \in Z'_{Max}} L^z$ will correspond to *one* index set $L^{2,0|h}$. We denote the h that

[159] Recall that a generic factor is a factor of the form $\nabla^{(m)}R_{ijkl}$ that is not contracting against any $\nabla\phi_h$.

corresponds to this *one* index set by α. We recall that the factor F_α is called the *critical factor*.

Now, for notational convenience we will be assuming that the free index ${}_{i_2}$ in each $C^{l,i_1i_2}_g, l \in L^{2,0}$ is a derivative index. This can be done with no loss of generality by virtue of the assumptions of Lemma 1.3 (no special free indices in any $C^{l,i_1i_2}_g$ and also $L^+_\mu = \emptyset$).

For any $a, b \le \tau$ we now denote by $L^{1,1|a,b}$ the index set of the 2-tensor fields indexed in $L^{1,1}$ with the additional feature that one free index belongs to F_a and the other to F_b. On the other hand for any $a \le \tau$, we denote by $L^{1,1|a,\tau+1}$ the index set of the 2-tensor fields where one index belongs to the factor F_a and the other belongs to a generic factor $\nabla^{(m)}R_{ijkl}$. Finally, we denote by $L^{1,1|\tau+1,\tau+1}$ the index of the 2-tensor fields where both free indices belong to (different) generic factors $\nabla^{(m)}R_{ijkl}$. Recall that we may assume wlog that if F_α or F_b are simple factors of the form $S_*\nabla^{(\rho)}R_{ijkl}$, then $L^{1,1|\alpha,b} = \emptyset$.

We distinguish two subcases: Either the critical factor F_α is in one of the forms $\nabla^{(A)}\Omega_h$, $S_*\nabla^{(\nu)}R_{ijkl}$, or it is in the form $\nabla^{(m)}R_{ijkl}$. We first consider the first subcase.

Proof of case $M = \mu = 2$ when the critical factor is in one of the forms $\nabla^{(A)}\Omega_h$, $S_*\nabla^{(\nu)}R_{ijkl}$:

We will firstly show that for each pair (α, b), (where $b \neq \alpha$ if $\alpha \le \tau + 1$ and neither F_α, F_b is a simple factor of the from $S_*\nabla^{(\nu)}R_{ijkl}$):

$$
\begin{aligned}
&\sum_{l \in L^{1,1|\alpha,b}} a_l C^{l,i_1i_2}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)\nabla_{i_1}\upsilon\nabla_{i_2}\upsilon + \\
&2\sum_{l \in L^{2,0|\alpha}} a_l \dot{C}^{l,i_1|i_*\to b}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)\nabla_{i_1}\upsilon\nabla_{i_*}\upsilon = \\
&Xdiv_{i_3}\sum_{h \in H} a_h C^{h,i_1i_2i_3}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)\nabla_{i_1}\upsilon\nabla_{i_2}\upsilon + \\
&\sum_{j \in J} a_j C^{j,i_1i_2}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)\nabla_{i_1}\upsilon\nabla_{i_2}\upsilon;
\end{aligned}
\tag{9.24}
$$

here $\dot{C}^{l,i_1|i_*\to b}_g$ stands for the tensor field that arises from $C^{l,i_1i_2}_g$ by erasing the (derivative) index ${}_{i_2}$ and adding a free derivative index onto F_b (and S_*-symmetrizing if needed). $\sum_{h\in H}\dots$ stands for a generic linear combination of acceptable 3-tensor fields with a u-simple character $\vec{\kappa}_{simp}$. $\sum_{j\in J}\dots$ stands for a linear combination of contractions that are simply subsequent to $\vec{\kappa}_{simp}$.

Note regarding the notation in (9.24): For each $b \le \tau$ the tensor field $\dot{C}^{l,i_1|i_*\to b}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)$ is well-defined. On the other hand, if $b = \tau + 1$ $\dot{C}^{l,i_1|i_*\to\tau+1}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)$ will be the sublinear combination in $\nabla_{i_*}[\dot{C}^{l,i_1}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)]$ where ∇_{i_*} is only allowed to hit one of the factors $\nabla^{(m)}R_{ijkl}$ which are not contracting against a factor $\nabla\phi_h$. Furthermore, if $\alpha = \tau + 1$ we additionally require that ∇_{i_*} can not hit the factor $\nabla^{(m)}R_{ijkl}$ to which ${}_{i_1}$ belongs.

We will show (9.24) below. For now, we show how (9.24) implies our claim.

Proof that (9.24) implies our claim (when the critical factor is in one of the forms $\nabla^{(A)}\Omega_h$, $S_*\nabla^{(\nu)}R_{ijkl}$): We make note of a straightforward corollary of (9.24): Making the $\nabla\upsilon$'s into $Xdiv$'s (using the last Lemma in the Appendix of [3]) and substituting this into our Lemma hypothesis, $L_g = 0$, we derive:

$$
\begin{aligned}
&\sum_{l\in L^{2,0|\alpha}} a_l Xdiv_{i_1} Xdiv_{i_2} C^{l,i_1 i_2}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)+\\
&\sum_{f\neq\alpha}\sum_{l\in L^{2,0|f}} a_l Xdiv_{i_1} Xdiv_{i_2} C^{l,i_1 i_2}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)+\\
&\sum_{l\in L^{2,0|a}} a_l Xdiv_{i_1} Xdiv_{i_*} \sum_{b\leq\tau+1,b\neq\alpha} \dot{C}^{l,i_1|i_*\to b}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)+\\
&\sum_{1\leq c<d\leq\tau+1,c,d\neq\alpha}\ \sum_{l\in L^{1,1|c,d}} a_l Xdiv_{i_1} Xdiv_{i_2} C^{l,i_1 i_2}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)+\\
&\sum_{h\in H} a_h Xdiv_{i_1}\dots Xdiv_{i_t} C^{h,i_1\dots i_t}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\\
&+\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)=0.
\end{aligned}
\tag{9.25}
$$

In particular, we have succeeded in replacing the old sublinear combinations $\sum_{l\in L^{1,1|\alpha,c}}\dots$ of 2-tensor fields with the two free indices belonging to different factors with new ones, which in particular arise from $\sum_{\in L^{2,0|\alpha}}$ in the precise way outlined above.

Now, we will be applying the grand conclusion to (9.25), making F_α the selected factor. We will be interested in the 2-tensor fields in the grand conclusion that will have the factor $\nabla\phi_{u+1}$ and the free index ${}_{i_1}$ contracting against/belonging to the crucial factor F_α.

Now, in order to apply the grand conclusion we must check that the extra claims are satisfied in this setting. We indeed have that $L^*_2=\emptyset$ by hypothesis and also that $L^+_2=\emptyset$, by inspection in (9.25). Then, we apply Lemma 9.1 to obtain an equation that is the same as (9.25) only with the additional restriction that the index set H is replaced by an index set H' which indexes terms which are contributors which are acceptable and have u-simple character $\vec{\kappa}_{simp}$ and $\nabla\phi_{u+1}$ not contracting against a special index. In view of this, we may now apply the grand conclusion to (9.25) and we obtain an equation:

$$
\begin{aligned}
&(2q_a+2\sigma_{a,*})\sum_{l\in L^{2,0|a}} a_l Xdiv_{i_2}C^{l,i_1i_2}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\
&\sum_{l\in\tilde{L}} a_l Xdiv_{i_2}C^{l,i_1i_2}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\
&\sum_{h\in H} a_h Xdiv_{i_2}\dots Xdiv_{i_t}C^{h,i_1\dots i_t}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\
&\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})=0.
\end{aligned}
\tag{9.26}
$$

Here q_α stands for the coefficient between parentheses in the grand conclusion (see the discussion *after* the "grand conclusion"–recall that the coefficient q_α depends on the *form* of the selected factor F_α) with $|I_1|=2$ for the factor F_α, while $\sigma_{\alpha,*}=-\sum_{v=1,v\neq\alpha}^{\sigma}2_v$ (recall the definition of 2_v from the discussion above the "grand conclusion"). Here all the 1-tensor fields are acceptable, and also have the factor $\nabla\phi_{u+1}$ contracting against the factor F_α. Moreover, the 2-tensor fields indexed in $\tilde{L}$ have the free index $_{i_2}$ *not* belonging to the selected factor F_α. (It will in fact be a non-dangerous index in some factor other than F_α). Terms indexed in H are contributors. Now, applying Lemma 9.1 if necessary,[160] we may also assume that $\nabla\phi_{u+1}$ is not contracting against a special index in F_α for any of the tensor fields indexed in H.[161]

Observe that $2(q_\alpha+\sigma_{\alpha,*})<0$ ($q_\alpha<0$ because $|I_1|=2$).

But then, just applying Corollary 1 in [6][162] to the above (and picking out the sublinear combination where $\nabla\phi_{u+1},\nabla\upsilon$ are contracting against the same factor), we derive an equation:

$$
\begin{aligned}
&(2q_\alpha+2\sigma_{\alpha,*})\sum_{l\in L^{2,0|\alpha}} a_l C^{l,i_1i_2}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon+\\
&\sum_{h\in H} a_h Xdiv_{i_3}\dots Xdiv_{i_t}C^{h,i_1\dots i_t}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\upsilon+\\
&\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1},\upsilon)=0.
\end{aligned}
\tag{9.27}
$$

Setting $\phi_{u+1}=\upsilon$ we derive our claim in this case $M=\mu=2$.

Proof of (9.24): We pick out some $b\neq\alpha$ and will show our claim for the index set $L^{1,1|(\alpha,b)}$. We distinguish two cases for the proof, based on the form of the factor F_α. Either F_α is of the form $\nabla^{(p)}\Omega_k$ or of the form $S_*\nabla^{(\nu)}R_{ijkl}$. We begin with the first case, which is the easiest.

[160] There is no danger of falling under a "forbidden case", by inspection.

[161] Recall that a special index is an internal index in some factor $\nabla^{(m)}R_{ijkl}$ or one of the indices $_{k},_{l}$ in some factor $S_*\nabla^{(\nu)}R_{ijkl}$.

[162] Notice that for the tensor fields of minimum rank there is a non-special free index.

First case: the factor F_α is in the form $\nabla^{(p)}\Omega_h$: We consider the equation $L_g(\Omega_1,\dots,\Omega_{k-1},\Omega_k\cdot\phi_{u+1},\dots\Omega_p,\phi_1,\dots,\phi_u)=0$ and we pick out the sublinear combination $T_g(\nabla\phi_{u+1})$ of complete contractions of length $\sigma+u+1$ with a factor $\nabla\phi_{u+1}$ contracting against F_b. Clearly, it follows that $T_g(\nabla\phi_{u+1})=0$, modulo complete contraction of length $\geq\sigma+u+2$. Moreover, we calculate:

$$
\begin{aligned}
&(0=)T_g(\nabla\phi_{u+1})=\sum_{l\in L^{1,1|\alpha,b}}a_l Xdiv_{i_1}C^{l,i_1i_2}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots\phi_u)\nabla_{i_2}\phi_{u+1}\\
&+2\sum_{l\in L^{2,0|\alpha}}a_l Xdiv_{i_1}\dot{C}^{l,i_1|i_*\to b}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots\phi_u)\nabla_{i_*}\phi_{u+1}\\
&+\sum_{l\in L'}a_l Xdiv_{i_2}C^{l,i_1i_2}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots\phi_u)\nabla_{i_1}\phi_{u+1}+\\
&\sum_{h\in H}a_h Xdiv_{i_2}\dots Xdiv_{i_t}C^{h,i_1\dots i_t}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots\phi_u)\nabla_{i_1}\phi_{u+1}\\
&+\sum_{j\in J}a_jC^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots\phi_{u+1})=0;
\end{aligned}
\tag{9.28}
$$

here the tensor fields indexed in L' have the free index ${}_{i_2}$ *not* belonging to the factor F_α and also have a u-simple character $\vec{\kappa}_{simp}$ and $\nabla\phi_{u+1}$ is not contracting against a special index in F_α. The tensor fields in H have a u-simple character $\vec{\kappa}_{simp}$ and have rank $t\geq 3$. The factor $\nabla\phi_{u+1}$ may be contracting against a special index in F_α, and in that case if $t=3$ then the other two free indices must be non-special. Also, we note that the tensor fields indexed in L',H potentially have one non-acceptable factor $\nabla\Omega_k$ (with only one derivative), and in that case if $t=3$ then the two free indices are not special. We call the tensor fields with a factor $\nabla\Omega_k$ or with the factor $\nabla\phi_{u+1}$ contracting against a special index "bad" tensor fields.

In the case where $\nabla\Omega_k$ is not contracting against a factor $\nabla\phi$, we may "get rid" of the bad tensor fields in (9.28) (modulo introducing tensor fields in the general form $\sum_{l\in L'}\dots$, $\sum_{h\in H}\dots$ which are not bad) via Corollary 2 in [6][163] or Lemma 4.7 in that paper. In the case where $\nabla\Omega_k$ is contracting against a factor $\nabla\phi$, we may "get rid" of the bad tensor fields in (9.28) (modulo introducing tensor fields in the general form $\sum_{l\in L'}\dots$, $\sum_{h\in H}\dots$ which are not bad) via Lemma 4.6 in [6].[164]

Therefore, we may assume wlog that all the tensor fields in (9.28) are acceptable and $\nabla\phi_{u+1}$ is not contracting against a special index. Therefore, this modified equation (9.28) shows (9.24) in the case $F_\alpha=\nabla^{(A)}\Omega_x$.

Proof of (9.24) when the crucial factor F_α is in the form $S_*\nabla^{(\nu)}R_{ijkl}$:

[163] This can be applied since the terms with minimum rank do not have special free indices, hence there is no danger of falling under a "forbidden case".

[164] This Lemma can be applied since the terms with minimum rank do not have special free indices, hence there is no danger of falling under a "forbidden case".

We define an operation $Link_{\alpha b}$ that acts on the complete contractions in L_g (recall that L_g stands for the LHS of the hypothesis of Lemma 1.3) by re-writing them in dimension $n+2$ and then hitting the factor F_α by a derivative ∇_c and the factor F_b by a derivative ∇^c. (As long as the weight of the complete contractions is $-n-2$, we will be considering the re-writing of our complete contractions in dimension $n+2$).

We now consider the $Image^{1,\beta}_{\phi_{u+1}}[L_g]$.[165] Recall the sublinear combinations $Image^{1,\beta,\sigma+u}_{\phi_{u+1}}[L_g]$, $Image^{1,\beta,\sigma+u+1}_{\phi_{u+1}}[L_g]$. We recall that in the sublinear combination of length $\sigma+u+1$ in $Image^{1,\beta}_{\phi_{u+1}}[L_g]$ we can still identify the factors F_α, F_b. On the other hand, in the sublinear combination of length $\sigma+u$ in $Image^{1,\beta}_{\phi_{u+1}}[L_g]$ we will now define the factors F_α, F_b: Recall that any complete contraction $C^t_g(\Omega_1,\dots,\Omega_p,\phi_{u+1},\phi_1,\dots,\phi_u)$ in this sublinear combination has arisen by replacing a (possibly symmetrized) curvature factor $\nabla^{(f)}R_{ijkl}$ by an expression $\nabla^{(f+2)}\phi_{u+1}\otimes g$, provided the two indices in g then contract against two indices in the same factor. Now, if the curvature factor that was replaced was *not* F_α or F_b, then we can straightforwardly identify F_α or F_b in C^t_g. If the curvature factor that was replaced was F_a, we will now define this new term $\nabla^{(p+2)}\phi_{u+1}$ to be the factor F_a. We use the same convention if the factor F_b was the curvature term that was replaced.

Thus, we can define the operation $Link_{a,b}[\dots]$ on the complete contractions in the sublinear combination $Image^{1,\beta}_{\phi_{u+1}}[L_g]$.

In view of the equation $Image^{1,\beta}_{\phi_{u+1}}[L_g]=0$ we derive an equation:

$$Link_{a,b}\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}-Image^{1,\beta}_{\phi_{u+1}}[Link_{a,b}\{L_g\}]=\sum_{w\in W}a_wC^w_g(\phi_{u+1}), \tag{9.29}$$

which holds modulo complete contractions of length $\geq\sigma+u+2$. The complete contractions C^w have length $\sigma+u+1$ and a factor $\nabla^{(p)}\phi_{u+1}$, $p\geq 2$. We observe that the left hand side of the above consists of complete contractions with length $\geq\sigma+u+1$. Thus, we can derive:

$$Link_{a,b}\{Image^{1,\beta}_{\phi_{u+1}}[L_g]\}-Image^{1,\beta}_{\phi_{u+1}}[Link_{a,b}\{L_g\}]=0, \tag{9.30}$$

modulo complete contractions of length $\geq\sigma+u+2$.

Moreover, any complete contraction of length $\sigma+u+1$ in the left hand side of the above will have a factor $\nabla\phi_{u+1}$ and an internal contraction involving a derivative index. We denote the left hand side by $Diff[L_g]$. We then define $Diff^*[L_g]$ the sublinear combination of complete contractions where $\nabla\phi_{u+1}$ is contracting against the factor F_α and the internal contraction belongs to the factor F_b. We then derive:

$$Diff^*[L_g]=0. \tag{9.31}$$

[165] Recall the definition of this sublinear combination from [8].

It is actually quite easy to understand how $Diff^*[L_g]$ arises from L_g:

Description of the linear combination $Diff^*[L_g]$: Let us write out L_g as a linear combination of complete contractions:

$$L_g = \sum_{t\in T} a_t C^t_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$$

Then, for each $t\in T$ we denote by Z^t the set of particular contractions $({}_x,{}_y)$ in C^t_g for which ${}_x$ belongs to F_α and ${}_y$ belongs to F_b. We formally define $Rep_{(x,y)}[C^t_g]$ to stand for the complete contraction that arises from C^t_g be erasing the contraction g^{xy} and making ${}_x$ contract against a factor $\nabla^x\phi_{u+1}$ and then adding a derivative index ∇^y onto the factor F_b to which the index ${}_y$ belongs (so now ${}_y$ is contracting against ∇^y and we have obtained an internal contraction).

We then calculate:

$$(0=)Diff^*[L_g] = \sum_{t\in T} a_t \sum_{(x,y)\in Z^t} Rep_{(x,y)}[C^t_g], \tag{9.32}$$

modulo complete contractions of length $\geq \sigma+u+2$. We also define $Rep^*_{(x,y)}[\dots]$ to stand for the operation that replaces the internal contraction $({}^y,{}_y)$ by an expression $(\nabla^y\omega,{}_y)$. By virtue of the operation Sub_ω (defined in the Appendix of [3]) applied to (9.32) we derive:

$$0 = \sum_{t\in T} a_t \sum_{(x,y)\in Z^l} Rep^*_{(x,y)}[C^t_g], \tag{9.33}$$

modulo complete contractions of length $\geq \sigma+u+3$.

Now in order to prove our claim (9.24), we define the operation $Hit^{\nabla\phi_{u+1}}_{F_\alpha}$ that acts on complete contractions and tensor fields by hitting the factor F_α by a derivative index ∇_i which we then contract against some factor $\nabla^i\phi_{u+1}$ (and in addition since F_α is in the form $S_*\nabla^{(\nu)}R_{ijkl}$ we S_*-symmetrize.

Furthermore, by construction we now observe that we can then write the right hand side of the above as:

$$
\begin{aligned}
&(0=)\sum_{t\in T} a_t \sum_{(x,y)\in Z^t} Rep^*_{(x,y)}[C^t_g] \\
&= \sum_{l\in L^{1,1|a,b}} a_l Xdiv_{i_1} Hit^{\nabla\phi_{u+1}}_{F_a} C^{l,i_1 i_2}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_2}\omega \\
&+2\sum_{l\in L^{2,0|a}} a_l Xdiv_{i_1} Hit^{\nabla\phi_{u+1}}_{F_a}[\dot{C}]^{l,i_1|i_*\to b}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_*}\omega+ \\
&\sum_{l\in\tilde{L}} a_l Xdiv_{i_3} C^{l,i_1 i_2 i_3}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\omega+ \\
&\sum_{h\in H} a_h Xdiv_{i_3}\dots Xdiv_{i_w} C^{h,i_1\dots i_w}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\omega+ \\
&\sum_{j\in J} a_j C^{j,i_1 i_2}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\omega;
\end{aligned}
\tag{9.34}
$$

here the tensor fields indexed in $\tilde{L}$ are all acceptable and have the feature that the free index ${}_{i_3}$ does *not* belong to F_α. Moreover, for each tensor field indexed in $\tilde{L}$ both the factors $\nabla\phi_{u+1}$, $\nabla\omega$ are *not* contracting against dangerous indices.[166]

The tensor fields indexed in H each have $w\ge 4$ and are necessarily acceptable. Moreover, either one or both of the factors $\nabla\phi_{u+1}$, $\nabla\omega$ may be contracting against special indices (if they are contracting against curvature factors).[167] Furthermore, the tensor fields indexed in J are u-subsequent to $\vec{\kappa}_{simp}$.

Now, we break up the index set H: We index in H_* the tensor fields that have the factor $\nabla\phi_{u+1}$ contracting against a special index in the factor $F_\alpha = S_*\nabla^{(\nu)}R_{ijkl}$. We want to derive an equation that will be precisely like (9.34), only with the extra restriction that $H_*=\emptyset$.

Proof that we may assume wlog that $H_*=\emptyset$ in (9.34): We may assume with no loss of generality (by just switching the last two indices in a curvature factor) that for each $h\in H_*$, $\nabla\phi_{u+1}$ is contracting against the index ${}_k$ in F_α. We then denote by

$$C^{h,i_2\dots i_w}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\hat{\phi}_{y_1},\dots\hat{\phi}_{y_{t-1}},\dots,\phi_u)$$

the tensor fields that arises from $C^{h,i_1\dots i_t}_g$ by replacing the expression

$$S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ijkl}\nabla^i\tilde{\phi}_h\nabla^k\phi_{u+1}\nabla^{r_1}\phi_{y_1}\dots\nabla^{r_t}\phi_{y_t}$$

by $\nabla^{(\nu+2-(t-1))}_{r_t\dots r_\nu jl}Y$ (recall that $t>0$). (Notice that we are obtaining complete contractions of weight $-n', n'\le n$, and moreover with $\sigma_1+\sigma_2=s-1$). We

[166]Recall that a dangerous index is either an internal index in some $\nabla^{(m)}R_{ijkl}$ or an index ${}_{k,l}$ in some $S_*\nabla^{(\nu)}R_{ijkl}$, or an index ${}_j$ in some S_*R_{ijkl}.

[167]This follows by construction, since these tensor fields arise from the tensor fields of minimum rank 2 in (1.6), all of whose free indices are non-special.

observe that all the tensor fields thus constructed will have the same $(u-(t-1))$-simple character, which we denote by $Cut(\vec{\kappa}_{simp})$. Then, we derive an equation:

$$\sum_{h\in H_*} a_h Xdiv_{i_3}\dots Xdiv_{i_t} C^{h,i_2\dots i_t}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\hat{\phi}_{y_1},\dots,\hat{\phi}_{y_{t-1}},\dots\phi_u)\nabla_{i_2}\omega$$
$$+\sum_{j\in J} a_j C^{j,i_2}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\hat{\phi}_{y_1},\dots,\hat{\phi}_{y_{t-1}},\dots\phi_u)\nabla_{i_2}\omega=0, \tag{9.35}$$

where each C^{j,i_2}_g is simply subsequent to $\vec{\kappa}_{simp}$.

Let $\tau\ge 2$ be the minimum rank of the tensor fields appearing above, and suppose they are indexed in $H_{*,\tau}$. Thus, (except for some special cases which we will treat momentarily), we apply our inductive assumption of Corollary 1 in [6] to the above;[168] and derive that for some linear combination of acceptable tensor fields (indexed in P below) with a simple character $Cut(\vec{\kappa}_{simp})$, so that:

$$\sum_{h\in H_{*,\tau}} a_h C^{h,i_2\dots i_{\tau-1}}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\hat{\phi}_{y_1},\dots\hat{\phi}_{y_{t-1}},\dots,\phi_u)\nabla_{i_2}\omega\nabla_{i_3}\upsilon\dots\nabla_{i_{\tau-1}}\upsilon-$$
$$\sum_{p\in P} a_p Xdiv_{i_\tau} C^{p,i_2\dots i_\tau}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\hat{\phi}_{y_1},\dots\hat{\phi}_{y_{t-1}},\dots,\phi_u)\nabla_{i_2}\omega\nabla_{i_3}\upsilon\dots\nabla_{i_{\tau-1}}\upsilon$$
$$=\sum_{j\in J} a_j C^{j,i_2\dots i_{\tau-1}}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\hat{\phi}_{y_1},\dots\hat{\phi}_{y_{t-1}},\dots,\phi_u)\nabla_{i_2}\omega\nabla_{i_3}\upsilon\dots\nabla_{i_\tau-1}\upsilon. \tag{9.36}$$

Now, we act on the above by an operation Op that formally replaces the expression $\nabla^A_{r_*r_1\dots r_A}\phi_{u+1}\nabla^{r_*}\phi_{y_t}$ by an expression

$$S_*\nabla^{A+t-1}_{r_1\dots r_{t-1}r_*\dots r_{A-2}}R_{ir_{A-1}kr_A}\nabla^i\tilde{\phi}_x\nabla^k\phi_{u+1}\nabla^{r_1}\phi_{y_1}\dots\nabla^{y_{t-1}}\phi_{y_{t-1}}\nabla^{r_*}\phi_{y_t}.$$

Thus, as explained in the proof that Lemma 3.1 in [6] implies Proposition 2.1 in [6], we derive a new equation:

$$\sum_{h\in H_{*,\tau}} a_h C^{h,i_1\dots i_{\tau-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\omega\nabla_{i_3}\upsilon\dots\nabla_{i_{\tau-1}}\upsilon=$$
$$\sum_{h\in H'_\tau} a_h C^{h,i_1\dots i_{\tau-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\omega\nabla_{i_3}\upsilon\dots\nabla_{i_{\tau-1}}\upsilon+$$
$$\sum_{p\in P} a_p Xdiv_{i_\tau} C^{p,i_2\dots i_\tau}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\omega\nabla_{i_3}\upsilon\dots\nabla_{i_{\tau-1}}\upsilon+$$
$$\sum_{j\in J} a_j C^{j,i_2\dots i_{\tau-1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\nabla_{i_2}\omega\nabla_{i_3}\upsilon\dots\nabla_{i_\tau-1}\upsilon, \tag{9.37}$$

[168] *Except* when there are "forbidden cases" appearing in the above–we will treat that case below. Note that if $\tau=2$ (9.35) can not fall under a forbidden case, since all free indices in the terms of minimum rank will be non-special.

where the tensor fields indexed in H'_τ are acceptable, have a u-simple character $\vec{\kappa}_{simp}$ and the factor $\nabla\phi_{u+1}$ is contracting against F_α, but not against a special index. The tensor fields indexed in J are simply subsequent to $\vec{\kappa}_{simp}$.

Now, replace the $\nabla\upsilon$'s by $Xdiv$s, and then replace into (9.34); iterating this step gives our desired conclusion (which is an equation in the form of (9.34) with $H_* = \emptyset$). If at that last step of this process we fall under a "forbidden case" of Corollary 1, we apply instead the "weaker version" of Corollary 1 from the Appendix in [6]. As we have noted above, $\tau > 2$ is we fall under "forbidden cases", hence the "weaker version" of Corollary 1 gives us our claim.

Therefore, we may now assume that each of the tensor fields indexed in H have the factor $\nabla\phi_{u+1}$ not contracting against a special index in F_α. $\Box$

Now our claim is only one step away: We apply the operation $Erase_{\phi_{u+1}}$ in (9.34) (since $H_* = \emptyset$ and F_α is a *non-simple* factor of the form $S_*\nabla^{(\nu)}R_{ijkl}$) this is well-defined). This is precisely our claim for (9.24).

Proof of our claim when the critical factor is of the form $\nabla^{(m)}R_{ijkl}$*:* An important observation: This hypothesis means that there are *no* 2-tensor fields in L_2 in our Lemma hypothesis with two free indices belonging to the same factor of the form $\nabla^{(B)}\Omega_h$ or $S_*\nabla^{(\nu)}R_{ijkl}$. This follows by the definition of the critical factor.

We denote by $F_1,\dots F_{\tau_1}$ the non-generic factors $\nabla^{(m)}R_{ijkl}$ in $\vec{\kappa}_{simp}$. We will then have that $L_2^{2,0} = \bigcup_{f=1}^{\tau_1} L_2^{2,0|f}\bigcup L_2^{2,0|\tau+1}$. Again denote by $L_2^{2,0|\alpha}$ the index set that corresponds to $\bigcup_{z\in Z'_{Max}} L^z$. Thus the critical factor will again be denoted by F_α (it will now be in the form $\nabla^{(m)}R_{ijkl}$). We first consider all the sets $L^{1,1|\alpha,b}$, $b = \tau_1+1,\dots b=\tau$ for which F_b is *not* in the form $\nabla^{(m)}R_{ijkl}$ (and is also *not* a simple factor of the form $S_*\nabla^{(\nu)}R_{ijkl}$). In that case, we claim that we can write:

$$\begin{aligned}
&Xdiv_{i_1}Xdiv_{i_2}\sum_{l\in L^{1,1|\alpha,b}} a_l C_g^{l,i_1i_2}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u) = \\
&-Xdiv_{i_1}Xdiv_{i_2}\sum_{l\in L^{2,0|\alpha}} a_l \dot{C}_g^{l,i_1|i_*\to b}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u) + \\
&\sum_{h\in H} a_h Xdiv_{i_1}\dots Xdiv_{i_t} C_g^{h,i_1\dots i_t}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u) \\
&+\sum_{j\in J} a_j C_g^j(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u),
\end{aligned} \tag{9.38}$$

where the tensor fields indexed in H u-simple character $\vec{\kappa}_{simp}$ and are acceptable and have $t \geq 3$. The complete contractions C^j are u-subsequent to $\vec{\kappa}_{simp}$. Note that if we can show the above, then by applying Lemma 10.2, we may also assume that they satisfy all the extra hypotheses of Lemma 1.3 pertaining to the extra claims.

Secondly, we consider the tensor fields indexed in $L^{1,1|\alpha,b}$ where F_b is in the form $\nabla^{(m)}R_{ijkl}$, where if $\alpha\in\{1,\dots,\tau_1\}$ then $b\neq\alpha$ (if $\alpha=\tau_1+1$, there are no restrictions). We claim that we can write:

$$\begin{aligned}
&Xdiv_{i_1}Xdiv_{i_2}\sum_{l\in L^{1,1|\alpha,b}}a_lC^{l,i_1i_2}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)=\\
&-Xdiv_{i_1}Xdiv_{i_2}\sum_{l\in L^{2,0|\alpha}}a_l\dot{C}^{l,i_1|i_*\to b}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\\
&-Xdiv_{i_1}Xdiv_{i_2}\sum_{l\in L^{2,0|b}}a_l\dot{C}^{l,i_1|i_*\to a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)+\\
&\sum_{h\in H}a_hXdiv_{i_1}\dots Xdiv_{i_t}C^{h,i_1\dots i_t}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\\
&+\sum_{j\in J}a_jC^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u),
\end{aligned}\tag{9.39}$$

with the same notational conventions as above.

Proof of (9.38) and (9.39): Both equations are easy to derive: We just consider the equation $Im^{1,\beta}_{\phi_{u+1}}[L_g]=0$ (see (7.61) and we pick out the sublinear combination where $\nabla\phi_{u+1}$ is contracting against F_α and $\nabla\omega$ against F_b. We then change both $\nabla\phi_{u+1}$ and $\nabla\omega$ into $Xdiv$s. $\Box$

Derivation of Lemma 1.3 in the case $M=\mu=2$ from the equations (9.38), (9.39) when the critical factor is of the form $\nabla^{(m)}R_{ijkl}$: Some notation: For each $\alpha\le\tau_1$ and each $b\le\tau_1$ with $b\neq\alpha$ or $b=\tau+1$, we denote by $C^{l,\{i_1i_2\}\to F_b}_g$ the tensor field that formally arises from $C^{l,i_1i_2}_g$, $l\in L^{2,0|\alpha}$ by erasing the two free indices $\nabla_{i_1},\nabla_{i_2}$ from F_α and adding them onto the factor F_b (if $b=\tau+1$ we add them onto any generic factor $\nabla^{(m)}R_{ijkl}$ and then add over all the tensor fields we can thus obtain). For $\alpha=\tau+1$ and $b\le\tau_1$ we define $C^{l,\{i_1i_2\}\to F_b}_g$ in the same way. If $\alpha=\tau+1,b=\tau+1$, we impose the extra restriction that the free indices are erased from the factor F_* to which they belong and then we add them to any other generic factor $\nabla^{(m)}R_{ijkl}$ other than F_*; we then add over all the tensor fields that we have obtained.

Now, for each $a\le\tau_1$ and also for $a=\tau+1$ we consider the grand conclusion with F_a being the selected factor. We denote by $\sum_{t\in T_a}a_tC^{t,i_1i_2}_g$ a generic linear combination of acceptable vector fields for which the free index ${}_{i_2}$ does not belong to F_a. We then derive an equation for each such a:

$$(-2q_a - 2(\sigma_1 - 2 + \sigma_2)) \sum_{l \in L_2^{2,0|a}} a_l X div_{i_2} C_g^{l,i_1 i_2}(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u) \nabla_{i_1} \phi_{u+1}$$
$$- 2 \sum_{b \in \{i, \dots, \tau_1, \tau+1\}} \sum_{l \in L_2^{2,0|b}} a_l X div_{i_2} C_g^{l,\{i_1 i_2\} \to F_a}(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u) \nabla_{i_1} \phi_{u+1}$$
$$+ \sum_{t \in T_a} a_t X div_{i_2} C_g^{t,i_1 i_2}(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u) \nabla_{i_1} \phi_{u+1} +$$
$$\sum_{h \in H} a_h X div_{i_2} \dots X div_{i_t} C_g^{h,i_1 \dots i_t}(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u) \nabla_{i_1} \phi_{u+1} +$$
$$\sum_{j \in J} a_j C_g^{j,i_1}(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u) \nabla_{i_1} \phi_{u+1}, \tag{9.40}$$

where each C^j is simply subsequent to $\vec{\kappa}_{simp}$ and each of the tensor fields indexed in H have $t \geq 3$ and are otherwise as in the conclusion of Lemma 1.3. The constant q_a is equal to $n - 2u - \mu - 2$ and is strictly positive (therefore observe that $-2q_a - 2(\sigma_1 - 2 + \sigma_2) < 0$). Therefore, by applying the inductive assumption of Corollary 1 in [6][169] we derive an equation, for each F_a in the form $\nabla^{(m)} R_{ijkl}$:

$$\sum_{l \in L_2^{2,0|a}} a_l C_g^{l,i_1 i_2}(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u) \nabla_{i_1} \phi_{u+1} \nabla_{i_2} \upsilon +$$
$$\frac{1}{q_a + 2(\sigma_1 - 2 + \sigma_2)} \sum_{b \in \{i, \dots, \tau_1, \tau+1\}} \sum_{l \in L_2^{2,0|b}} a_l C_g^{l,\{i_1 i_2\} \to F_a}(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)$$
$$\nabla_{i_1} \phi_{u+1} \nabla_{i_2} \upsilon + \sum_{h \in H} a_h X div_{i_3} C_g^{h,i_1 \dots i_t}(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u) \nabla_{i_1} \phi_{u+1} \nabla_{i_2} \upsilon$$
$$= \sum_{j \in J} a_j C_g^{j,i_1}(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u) \nabla_{i_1} \phi_{u+1}. \tag{9.41}$$

We are therefore reduced to proving our claim under the assumption that the sublinear combination $\sum_{l \in L_2^{2,0}} a_l C_g^{l,i_1 i_2}$ in our Lemma hypothesis can be expressed in the form:

$$\nabla^{2,spread}_{i_1 i_2} [\sum_{b \in B} a_b C_g^b(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)],$$

where the complete contractions C_g^b are in the form (1.5), have weight $-n+4$, are acceptable of simple character $\vec{\kappa}_{simp}$. The symbol $\nabla^{2,spread}_{i_1 i_2}$ means that we

[169] The terms of minimum rank above all involve terms with non-special free indices, hence there is no danger of falling under a "forbidden case". Also, we fall under the inductive assumption of that Lemma because we increased the number of factors $\nabla \phi_h$, while keeping all the other parameters of the induction fixed.

may hit any factor of the form $\nabla^{(m)}R_{ijkl}$, and then we add over all these factors we have hit.

Therefore, applying (9.38) and (9.39) to this setting we may in addition assume that the sublinear combination of 2-tensor fields in (1.6) where one free index ${}_{i_1}$ belongs to a factor $\nabla^{(m)}R_{ijkl}$ and one to a factor $S_*\nabla^{(\nu)}R_{ijkl}$ is:

$$-\nabla^{2|i_1\to\nabla^{(m)}R_{ijkl},i_2\to S_*\nabla^{(\nu)}R_{ijkl}}[\sum_{b\in B}a_bC^b_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)],$$

where the symbol outside brackets means that we are considering the sublinear combination in $\nabla^{(2)}_{i_1i_2}$ where ${}_{i_1}$ is forced to hit a factor $\nabla^{(m)}R_{ijkl}$ and ${}_{i_2}$ is forced to hit a factor $S_*\nabla^{(\nu)}R_{ijkl}$ (and we can analogously obtain a new true equation for the sublinear combination of terms where one free index belongs to a factor $\nabla^{(m)}R_{ijkl}$ and the other to a simple factor of the form $\nabla^{(y)}\Omega_h$).

Moreover, applying (9.39) and making $\nabla\phi_{u+1}$, $\nabla\upsilon$ into $Xdiv$s, (by virtue of the last Lemma in the Appendix of [3]) we may assume that the sublinear combination of 2-tensor fields in (1.6) where both free indices ${}_{i_1},{}_{i_2}$ belong to (different) factors $\nabla^{(m)}R_{ijkl}$ is:

$$-2\nabla^{2|i_1\to\nabla^{(m)}R_{ijkl},i_2\to\nabla^{(m)}R_{ijkl}}[\sum_{b\in B}a_bC^b_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)],$$

where the symbol outside brackets means that we are considering the sublinear combination in $\nabla^{i_1i_2}$ where ${}_{i_1},{}_{i_2}$ are forced to hit two different factors of the form $\nabla^{(m)}R_{ijkl}$.

Now pick any $A\in\{1,\dots,\tau_1+1\}$. Applying the grand conclusion with F_A being the selected factor, we derive:

$$\begin{aligned}&(-2q_1-2(\sigma_1-1)-2\sigma_2)\sum_{b\in B}a_bXdiv_{i_2}\nabla^{2,\{i_1,i_2\}\to F_1}_{i_1i_2}C^b_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\\&\nabla_{i_1}\phi_{u+1}+\sum_{t\in T}a_tXdiv_{i_2}C^{t,i_1i_2}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\&\sum_{h\in H}a_hXdiv_{i_2}\dots Xdiv_{i_t}C^{h,i_1\dots i_t}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\&\sum_{j\in J}a_jC^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1})=0.\end{aligned}\tag{9.42}$$

The symbol in the first line means that we hit the factor F_A with two derivatives $\nabla_{i_1i_2}$. Here the tensor fields indexed in T have ${}_{i_2}$ *not* belonging to F_A. Since the coefficient in the first line is non-zero, we may apply our inductive assumption of Corollary 1 in [6],[170] pick out the sublinear combination where $\nabla\upsilon$ is contracting against F_1 and then set $\phi_{u+1}=\upsilon$. This is our desired conclusion. □

[170] The terms of minimum rank above contain no special free indices, hence there is no danger of falling under a "forbidden case".

9.7 Proof of Lemma 1.3 in the subcase $\mu = 1$.

We proceed to show the remaining case for Lemma 1.3, the case $\mu = 1$. Recall that by Lemma 10.6 we may assume that there are to 1-tensor fields in our Lemma hypothesis for which the free index belongs to a factor $S_*\nabla^{(\nu)}R_{ijkl}$.

Special cases: We single out certain special cases which will be treated in a Mini-Appendix at the end of this paper. The "special subcases" are when $\sigma_2 > 0$ in $\vec{\kappa}_{simp}$,[171] and the terms of maximal refined double character in (1.6) *either* have no removable index,[172] *or* the refined double characters correspond to the form:

$$\begin{aligned} &contr(\nabla_{(free)}R_{\sharp\sharp\sharp\sharp} \otimes R_{\sharp\sharp\sharp\sharp} \otimes \cdots \otimes R_{\sharp\sharp\sharp\sharp} \otimes \\ &S_*R_{i\sharp\sharp\sharp} \otimes \cdots \otimes S_*R_{i\sharp\sharp\sharp} \otimes \nabla^{(2)}_{y\sharp}\Omega_1 \otimes \cdots \otimes \nabla^{(2)}_{y\sharp}\Omega_p \otimes \nabla\phi_1 \otimes \cdots \otimes \nabla\phi_u). \end{aligned} \tag{9.43}$$

(In the above, each index $_\sharp$ must contract against another index in the form $_\sharp$; the indices $_y$ are either contracting against indices $_\sharp, _y$ or contract against a factor $\nabla\phi_h$). We remark that in the rest of this proof we will be assuming that the terms of maximal refined double character in (1.6) are *not* in the form (9.43) with $\sigma_2 > 0$.[173]

In this case the different refined double characters of the vector fields $C^{l,i_1}_g, l \in L_1$ are fully characterized by specifying the factor in $\vec{\kappa}_{simp}$ to which $_{i_1}$ belongs. Recall the discussion on the index sets $L^z, z \in Z'_{Max}$. Observe that in this case the index set Z'_{Max} will consist of one element, and the sublinear combination stands for the sublinear combination of 1-tensor fields in the LHS of the Lemma hypothesis for which the free index $_{i_1}$ belongs to the *critical(=crucial) factor*.[174] Denote by F_α the crucial factor.

Again, we start with the case where the critical factor F_α is of the form $\nabla^{(B)}\Omega_h$.

Proof of the claim when the critical factor is in the form $\nabla^{(B)}\Omega_h$: We denote the index set of the 1-tensor fields in the Lemma hypothesis where $_{i_1}$ belongs to the crucial factor by L^α_1. (In other words $\bigcup_{z \in Z'_{Max}} L^z = L^\alpha_1$).

We will then initially be using an analogue of (9.24). We introduce some notation to serve our purposes.

Notation: For each $l \in L_1$, we denote by $C^l_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)$ the complete contraction (of weight $-n+2$) that arises from $C^{l,i_1}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u)$ by erasing the (derivative) index $_{i_1}$. For each $x \le \sigma_1$, we then define

[171] We recall that σ_2 stands for the number of factors $S_*\nabla^{(\nu)}R_{ijkl}$ in $\vec{\kappa}_{simp}$.

[172] In particular, all their factors must be in the form R_{ijkl}, S_*R_{ijkl} without derivatives, *or* in the form $\nabla^{(2)}\Omega_h$.

[173] We are again applying the convention that we do not write out any derivative indices in a factor $\nabla^{(m)}R_{ijkl}$ that contract against factors $\nabla\phi$).

[174] Recall that given the simple character $\vec{\kappa}_{simp}$ of the tensor fields appearing in the hypothesis of our Lemma, the crucial factor is either a well-defined factor in one of the forms $\nabla^{(B)}\Omega_h$, $\nabla^{(m)}R_{ijkl}$, $S_*\nabla^{(\nu)}R_{ijkl}$ *or* the generic set of factors $\nabla^{(m)}R_{ijkl}$ that are not contracting against any factor $\nabla\phi_h$.

$Pass_x^{i_1}[C^l_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]$ as follows: If $x\le\sigma_1$, it will stand for the vector field that arises from C^l_g by hitting the x^{th} factor $\nabla^{(m)}R_{ijkl}$ by a derivative index ∇_{i_1}.

For each $x\le\sigma_1$, we denote by $L^x_1\subset L_1$ the index set of the vector fields in our Lemma hypothesis for which ${}_{i_1}$ belongs to the x^{th} factor. We denote by $\vec{\kappa}_x$ the $(u+1)$-simple character that arises by contracting the free index ${}_{i_1}$ against a factor $\nabla\phi_{u+1}$.

We will prove that for each $x, 1\le x\le\sigma_1$:

$$\begin{aligned}&\sum_{l\in L^x_1} a_l C^{l,i_1}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}\\&+\sum_{l\in L^\alpha_1} a_l Pass_x^{i_1}C^{l,i_1}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\&\sum_{h\in H^x} a_h Xdiv_{i_2}C^{h,i_1i_2}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\&\sum_{j\in J} a_j C^{j,i_1}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}=0,\end{aligned}\tag{9.44}$$

modulo complete contractions of length $\ge\sigma+u+2$. Each C^{j,i_1}_g is u-subsequent to $\vec{\kappa}_{simp}$, each vector field $C^{h,i_1i_2}_g$ has a u-simple character $\vec{\kappa}_{simp}$, $(u+1)$-weak character $Weak(\vec{\kappa}_x)$ and is either acceptable or has one unacceptable factor $\nabla\Omega_h$. We will discuss how (9.44) follows below. Let us now check how it applies to show our Lemma.

Proof that (9.44) implies case B of Lemma 1.3 in this case $\mu=1$ when the critical factor is of the form $\nabla^{(p)}\Omega_h$:

We make the factor $\nabla\phi_{u+1}$ into an $Xdiv$ in each of the above equations (appealing to the last Lemma in the Appendix of [3]), and then we replace the sublinear combination

$$\sum_{x=1}^{\sigma_1} Xdiv_{i_1}\sum_{l\in L^x_1} a_l C^{l,i_1}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$$

in our Lemma hypothesis by

$$\begin{aligned}&-\sum_{x=1}^{\sigma_1}\sum_{l\in L^\alpha_1} a_l Xdiv_{i_1}Pass_x^{i_1}C^{l,i_1}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)+\\&\sum_{h\in H} a_h Xdiv_{i_1}Xdiv_{i_2}C^{h,i_1i_2}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)+\\&\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u).\end{aligned}\tag{9.45}$$

Thus, we obtain an equation:

$$
\begin{aligned}
&\sum_{l\in L_1^\alpha} a_l Xdiv_{i_1} C_g^{l,i_1}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\\
&-\sum_{x=1}^{\sigma_1}\sum_{l\in L_1^\alpha} a_l Xdiv_{i_1} Pass_x^{i_1} C_g^{l,i_1}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)+\\
&\sum_{l\in \tilde{L}_1} a_l Xdiv_{i_1} C_g^{l,i_1}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)+\\
&\sum_{h\in H} a_h Xdiv_{i_1} Xdiv_{i_2} C_g^{h,i_1 i_2}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)+\\
&\sum_{j\in J} a_j C_g^{j}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u).
\end{aligned}
\tag{9.46}
$$

The vector fields indexed in $\tilde{L}_1$ are acceptable but ${}_{i_1}$ belongs to some non-crucial factor in the form $\nabla^{(B)}\Omega_h$.

Again by inspection we have that $L_1^+=\emptyset$. By applying Lemma 9.1, we may assume wlog that the tensor fields indexed in H are all acceptable. We also apply Lemmas 10.1, 10.2 if necessary to ensure that the above fulfills the requirements to apply the "grand conclusion". We may then apply the grand conclusion to the above, making F_1 the selected factor. We derive an equation:

$$
\begin{aligned}
&\sum_{l\in L_1^\alpha} a_l(-q_1-\sum_{x=1,x\neq\alpha}^{\sigma_1}\overline{2}_x) C_g^{l,i_1}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\
&\sum_{h\in H} a_h Xdiv_{i_2} C_g^{h,i_1 i_2}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+\\
&\sum_{j\in J} a_j C_g^{j}(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1});
\end{aligned}
\tag{9.47}
$$

here the tensor fields indexed in H have a u-simple character $\vec{\kappa}_{simp}$, a weak $(u+1)$-character $Weak(\vec{\kappa}^+_{simp})$ and possibly one un-acceptable factor $\nabla\Omega_h$ (q_1 stands for the coefficient inside parentheses in the first line of the grand conclusion, and it depends on the form of the selected factor). Using Lemma 9.1 if necessary we may assume that all terms in H are acceptable with a $(u+1)$-simple character $\vec{\kappa}^+_{simp}$.

Now, under these assumptions, notice that if $(-q_1-\sum_{x=1,x\neq a}^{\sigma_1}2_x)\neq 0$ then dividing by this number we derive our claim. Let us now derive our claim in the case where $(-q_1-\sum_{x=1,x\neq a}^{\sigma_1}2_x)=0$.

The remaining case: Notice that the only case in which the number in parentheses is zero is when we have $\sigma_1=\sigma_2=0$ and all factors $\nabla^{(B)}\Omega_h$ have all their non-free indices contracting against factors $\nabla\phi_h$. We also observe that furthermore there can only be one free index among all the tensor fields in our

induction hypothesis (in other words $L_{>1} = \emptyset$). In that case we just read off our claim directly from (1.6) as follows: We break L_1 into sets $L_1^h, h = 1, \dots, p$ depending on which factor $\nabla^{(B)}\Omega_h$ contains the free index. Then, for each pair $1 \le h_1 < h_2 \le p$ we pick out the sublinear combination in (1.6) where the one contraction not involving factors $\nabla\phi_x$ is between $\nabla^{(B)}\Omega_{h_1}, \nabla^{(C)}\Omega_{h_2}$. This sublinear combination must vanish separately. We then erase the two (derivative) indices involved in these complete contractions and we derive:

$$\{\sum_{l \in L_1^{h_1}} a_l + \sum_{l \in L_2^{h_2}} a_l\} C^l_g \tag{9.48}$$

This then implies that $\sum_{l \in L_1^h} a_l = 0$ for every $h = 1, \dots, p$, since $p = \sigma \ge 3$.

So, we only have to show (9.44). But this follows by the exact same argument we used to prove equation (9.24) in the case $\mu = 2$: We replace factor crucial factor $\nabla^{(B)}\Omega_h$ by $\nabla^{(B)}(\Omega_h \cdot \omega)$ in our Lemma hypothesis and then pick out the sublinear combination with a factor $\nabla\omega$ contracting against the x^{th} factor $\nabla^{(m)}R_{ijkl}$. This sublinear combination must vanish separately and this is precisely our claim, (9.44). □

Proof of the claim when the crucial factor is of the form $\nabla^{(m)}R_{ijkl}$: Again, by the definition of the critical(=crucial) factor, it follows that no vector field indexed in L_1 will have its free index belonging to a factor $\nabla^{(B)}\Omega_h$ or $S_*\nabla^{(\nu)}R_{ijkl}$.

We then repeat the same argument as in the previous case: Let us denote by $L_1^1, \dots L_1^{\tau_1}, L_1^{\tau_1+1}$ all the index sets that correspond to the various factors $F_1, \dots, F_{\tau_1}, F_{\tau+1}$[175] to which the free index may belong. Recall that q_1 will stand for the coefficient in the first line of the grand conclusion; recall $q_1 \ge 0$. We distinguish two cases: Either $q_1 = 0$ or $q_1 > 0$ (recall q_1 is the coefficient between parentheses in the grand conclusion, in this case $q_1 = n - 2u - \mu - 3$). We first prove our claim in the first (very special) case.

Subcase $q_1 = 0$: In that case we clearly have that $\sigma_1 = 1$ and all the indices in each factor $\nabla^{(B)}\Omega_h$ are contracting against a factor $\nabla\phi_y$, and also all the indices $_{r_1, \dots, r_\nu, j}$ in each factor $S_*\nabla^{(\nu)}R_{ijkl}$ are contracting against some factor $\nabla\phi'_h$. Furthermore, the crucial factor must be in the form $\nabla^{(m)}_{r_1 \dots r_m}\nabla_{i_1}R_{ijkl}$ (where $_{r_1, \dots, r_m}$ are contracting against factors $\nabla\phi_x$). So we can write:

$$\sum_{l \in L'_1} a_l C^{l,i_1}_g = \nabla^{cruc}_{i_1}[\sum_{b \in B} a_b C^b_g]$$

modulo longer complete contractions, where each C^b_g is an acceptable complete contraction of weight $-n+2$ with simple character $\vec{\kappa}_{simp}$ and $\nabla^{cruc}_{i_1}$ means that the derivative ∇_{i_1} is forced to hit the (unique) crucial factor $\nabla^{(m)}R_{ijkl}$.

[175] Recall that $F_1, \dots F_{\tau_1}$ stand for the factors $\nabla^{(m)}R_{ijkl}$ in $\vec{\kappa}'_{simp}$ that are contracting against some factor $\nabla\phi_h$. $F_{\tau+1}$ stands for the set of factors $\nabla^{(m)}R_{ijkl}$ in $\vec{\kappa}_{simp}$ that are not contracting against any factors $\nabla\phi_h$.

We then pick any factor $F_c \neq F_1$ (F_1 is the crucial factor) and we apply the grand conclusion to L_g with F_c being the selected factor. We obtain an equation:

$$\sum_{b\in B} a_b 2\nabla^{F_c}_{i_1}[C^b_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\nabla^{i_1}\phi_{u+1}+ \\ \sum_{h\in H} a_h Xdiv_{i_2}\dots Xdiv_{i_a} C^{h,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla^{i_1}\phi_{u+1}+ \\ \sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)=0, \tag{9.49}$$

modulo complete contractions of length $\geq \sigma+u+2$. Here the tensor fields indexed in H have $a\geq 2$ and have a u-simple character $\vec{\kappa}_{simp}$ and the factor $\nabla\phi_{u+1}$ contracting against F_c. They may also have one unacceptable factor $\nabla\Omega_x$.

Now, by applying Lemma 9.1,[176] we may assume wlog that for each tensor field indexed in H above there are no unacceptable factors and the factor $\nabla\phi_{u+1}$ are contracting against a non-special index.

We then apply $Erase_{\phi_{u+1}}$ to the above and then add a derivative index ∇_{i_*} onto the crucial factor F_1, which we then contract against a factor $\nabla_{i_*}\phi_{u+1}$. This gives us an equation:

$$\sum_{b\in B} a_b 2\nabla^{cruc}_{i_1}[C^b_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\nabla^{i_1}\phi_{u+1}+ \\ \sum_{h\in H'} a_h Xdiv_{i_2}\dots Xdiv_{i_a} C^{h,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla^{i_1}\phi_{u+1}+ \\ \sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)=0, \tag{9.50}$$

where the tensor fields indexed in H' have a u-simple character $\vec{\kappa}_{simp}$ and are acceptable, and $\nabla\phi_{u+1}$ is contracting against a non-special index in the crucial factor, by construction. This is our desired conclusion in this case.

Subcase $q_1>0$: Observe that we will either have $q_1>2(\sigma_1-1)$ or $q_1=2(\sigma_1-1)$.[177] For both cases the starting point will be the same:

We pick any $c\in\{1,\dots,\tau_1,\tau_1+1\}$ and we consider the grand conclusion where we make F_c the selected factor.[178] We derive:

[176] By inspection, the above does not fall under a forbidden case of Lemma 9.1.

[177] Notice (by a counting argument) that if the 1-tensor fields of maximal refined double character in (1.6) are in the form (9.43), then $q_1=2(\sigma_1-1)$.

[178] Recall that F_c is always a factor $\nabla^{(m)}R_{ijkl}$.

$$\sum_{l\in L_1^c} a_l(-q_1-2)C^{l,i_1}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+$$
$$2\sum_{b\in\{1,\dots,\tau_1,\tau_1+1\}}\sum_{l\in L_1^b} a_l C^{l,i_1\to F_c}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}+$$
$$(\sum_{b\in B'} a_b C^{b,i_1}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}) \tag{9.51}$$
$$+\sum_{h\in H} a_l Xdiv_{i_2}\dots Xdiv_{i_y}C^{h,i_1\dots i_y}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}=$$
$$\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),$$

where the tensor fields indexed in H and complete contractions indexed in J are as in the conclusion of the grand conclusion. The linear combination $\sum_{b\in B'}\dots$ (defined in Definition 5.2) arises *only* when the 1-tensor fields of maximal refined double character in (1.6) are in the form (9.43). Applying Lemma 9.1 if necessary,[179] we may assume the tensor fields in H are acceptable.

Now, we divide by $(-q_1-2)$, and we make $\nabla\phi_{u+1}$ into an $Xdiv$ and then replace into our Lemma hypothesis, (1.6). We see that we are reduced to proving our Lemma 1.3 in this case with the sublinear combination $\sum_{l\in L_1} a_l\dots$ replaced by a new sublinear combination $\sum_{l\in \underline{L}_1} a_l\dots$ with certain special features:

Special features: Let us divide $\underline{L}_1$ as before into $\underline{L}_1^c$, $c\in\{1,\dots,\tau_1,\tau_1+1\}$. Then the sublinear combinations $\sum_{l\in\underline{L}_1^c}\dots$ are related as follows: There exists a linear combination of acceptable complete contractions

$$\sum_{t\in T} a_t C^t_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u),$$

with length $\sigma+u$, weight $-n+2$ and u-simple character $\vec{\kappa}_{simp}$ so that for each $c\in\{1,\dots,\tau_1,\tau+1\}$:

$$\sum_{l\in\underline{L}_1^c} a_l C^{l,i_1}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)=\nabla^{i_1}_{F^c}\sum_{t\in T} a_t C^t_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u), \tag{9.52}$$

where $\nabla^{i_1}_{F^c}$ means that ∇^{i_1} is forced to hit the factor F^c.

Then, applying the grand conclusion with F^c being the selected factor, we derive:

[179] The Lemma can be applied, since we are assuming that we did not start out with the "special cases", described in the beginning of our subsection.

$$
\begin{aligned}
&(-q_1+2(\sigma_1-1))\sum_{t\in T}a_t\nabla^{i_1}_{F^c}[C^t_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\nabla_{i_1}\phi_{u+1}+\\
&\sum_{h\in H}a_lXdiv_{i_2}\dots Xdiv_{i_y}C^{h,i_1\dots i_y}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}=\\
&\sum_{j\in J}a_jC^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}).
\end{aligned}
\tag{9.53}
$$

Thus, if $q_1>2(\sigma_1-1)$, since the quantity in parentheses is non-zero, we only have to apply Lemma 9.1 to ensure that all tensor fields indexed in H are acceptable and we are done.

If $q_1=2(\sigma_1-1)$, we distinguish two further subcases: Either $\sigma_2+p>0$ or $\sigma_2=p=0$.

In the first case, it again follows (from the definition of q_1 and a counting argument) that for each vector field in our Lemma hypothesis, the factors $S_*\nabla^{(\nu)}_{r_1\dots r_\nu}R_{ijkl}$ must have all their indices ${}_{r_1},\dots,{}_{r_\nu},{}_j$ contracting against factors $\nabla\phi'_h$ and all factors $\nabla^{(B)}\Omega_x$ must have at least two of their indices contracting against factors $\nabla\phi_h$.

Let us consider the first subcase: Making a factor $F_d\neq\nabla^{(m)}R_{ijkl}$ into the selected factor and applying the grand conclusion we derive an equation:

$$
\begin{aligned}
&2\sigma_1\nabla^{i_1}_{F^d}[\sum_{t\in T}a_tC^t_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\nabla_{i_1}\phi_{u+1}+\\
&\sum_{h\in H}a_lXdiv_{i_2}\dots Xdiv_{i_y}C^{h,i_1\dots i_y}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}=\\
&\sum_{j\in J}a_jC^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}),
\end{aligned}
\tag{9.54}
$$

where the tensor fields indexed in H have $\nabla\phi_{u+1}$ contracting against the selected factor F_c. As usual, they may have one unacceptable factor $\nabla\Omega_x$ but then $\nabla\phi_{u+1}$ will not be contracting against a special index.

Now, by applying Lemma 9.1 if necessary, we may assume that all tensor fields indexed in H are acceptable and that $\nabla\phi_{u+1}$ is not contracting against a special index in any of the tensor fields in (9.54).

Under all the assumptions above, we apply the eraser to $\nabla\phi_{u+1}$ to the above (notice that this can be done and produces acceptable tensor fields, by virtue of our assumptions) and then adding a contracted derivative index ∇^{i_1} onto the crucial factor $F_1=\nabla^{(m)}R_{ijkl}$, and then contacting against a factor $\nabla_{i_1}\phi_{u+1}$, we obtain:

$$2\sum_{t\in T} a_t\nabla^{i_1}_{F^1}[C^t_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)]\nabla_{i_1}\phi_{u+1}+$$
$$\sum_{h\in H'} a_l Xdiv_{i_2}\dots Xdiv_{i_y} C^{h,i_1\to F_1,i_2\dots i_y}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\phi_{u+1}=$$
$$\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_{u+1}). \tag{9.55}$$

This is our desired conclusion, in the case $\sigma_2+p>0$.

Finally, the case where $\sigma_2=p=0$, $q_1=2(\sigma_1-1)$. Notice that necessarily then, in our Lemma hypothesis there are no complete contractions C^j_g. Recall (9.52), which still holds in this case.

Since $q_1=2(\sigma_1-1)$ it follows that each C^t_g has the property that in *each* factor $\nabla^{(m)}R_{ijkl}$ *all* the derivative indices must be contracting against a factor $\nabla\phi_h$. Therefore, each complete contraction in our Lemma hypothesis[180] must have precisely two derivative indices among all the factors $\nabla^{(m)}R_{ijkl}$ that are not contracting against a factor $\nabla\phi_x$ (this follows by weight considerations).

Also, the only tensor fields appearing in (1.6), other than the ones indexed in L_1 can have rank 2 (recall that this sublinear combination of these 2-tensor fields is denoted by $\sum_{l\in L_2} a_l C^{l,i_1i_2}_g$), and they must be in one of two special forms:

1. *Either* $C^{l,i_1i_2}_g$ will have two factors $\nabla^{(m)}R_{ijkl}$ with the indices ${}_i$ being free and all derivative indices contracting against some factor $\nabla\phi_x$, and furthermore all the other $\sigma-2$ factors $\nabla^{(m)}R_{ijkl}$ must have no free indices and each derivative index contracting against some factor $\nabla\phi_h$.
2. *Or* $C^{l,i_1i_2}_g$ will have both free indices being internal non-antisymmetric indices in some factor $\nabla^{(m)}R_{ijkl}$, and all derivative indices in all the other $\sigma-1$ factors $\nabla^{(m)}R_{ijkl}$ are contracting against factor $\nabla\phi_x$.

Let us prove the claim in this case: Denote by $L_{2,diff}\subset L_2$ the index set of 2-tensor fields where the two free indices ${}_{i_1},{}_{i_2}$ belong to different factors. Let $C^{l,i_1i_2}_g(\phi_1,\dots,\phi_u)g_{i_1i_2}$ be the complete contraction that arises from $C^{l,i_1i_2}_g$ by contracting the two indices ${}_{i_1},{}_{i_2}$ against each other.

Now, given a complete contraction $C^t_g(\phi_1,\dots,\phi_u)$ in the form (1.5) with a u-simple character $\vec{\kappa}_{simp}$, we denote by $\tau[t]$ the number of pairs of factors (T_a,T_b) with at least one particular contraction between them.

We claim:

$$\sum_{l\in L_{2,diff}} a_l C^{l,i_1i_2}_g(\phi_1,\dots,\phi_u)g_{i_1i_2}=\sum_{t\in T} a_t\tau[t]C^t_g(\phi_1,\dots,\phi_u). \tag{9.56}$$

[180]In other words, we momentarily think of each $Xdiv$ in our Lemma hypothesis as a sum of complete contractions.

Let us check how the above implies our claim:

Proof that (9.56) implies our claim: We will prove this inductively: Let M be the maximum value of $\tau[t], t \in T$. Clearly $M > 0$ (simply because the exist a pair of indices in two different factors that contract against each other, by construction). Denote by $T_M \subset T$ the corresponding index set. We will then prove:

$$\sum_{t\in T_M} a_t \nabla_{i_1} C^t_g \nabla^{i_1}\phi_{u+1} = \sum_{h\in H} a_h X div_{i_2} C^{h,i_1 i_2}_g \nabla_{i_1}\phi_{u+1} + \sum_{t\in T'} a_t \nabla_{i_1} C^t_g \nabla^{i_1}\phi_{u+1}; \tag{9.57}$$

here the terms indexed in T' are generic complete contractions in the form (1.5) with a u-simple character $\vec{\kappa}_{simp}$ and weight $-n+2$ and moreover $\tau[t] < M$. The 2-tensor fields in $\sum_{h\in H} a_h \dots$ are as described in the claim of Lemma 1.3. If we can prove this, then by replacing the $\nabla\phi_{u+1}$ by an $Xdiv$,[181] replacing back into our Lemma hypothesis and then iterating this step at most four times, we derive our claim.

Thus, matters are reduced to proving (9.57):

Proof of (9.57): Refer to equation (6.25). In view of (9.56), we derive an equation:

$$\sum_{t\in T_M} M\cdot a_t \nabla_{i_1} C^t_g \nabla^{i_1}\phi_{u+1} = \sum_{h\in H} a_h X div_{i_2} C^{h,i_1 i_2}_g - \sum_{t\in T\setminus T_M} a_t\cdot\tau[t] \nabla_{i_1} C^t_g \nabla^{i_1}\phi_{u+1}. \tag{9.58}$$

Thus, dividing by M in the above, we derive (9.57). $\Box$

Proof of (9.56): For each $C^{l,i_1 i_2}_g, l \in L_{2,diff}$, let us denote by T_{i_1}, T_{i_2} the two factors $\nabla^{(m)}R_{ijkl}, \nabla^{(m'}R_{i'j'k'l'}$ to which the indices ${}_{i_1}, {}_{i_2}$ belong.

Definition 9.3 *We assume wlog that they occupy the positions ${}_{i}, {}_{i'}$. Let us denote by $L^\alpha_{2,diff}$, $L^\beta_{2,diff}$, $L^\gamma_{2,diff}$, $L^\delta_{2,diff}$ the index sets of tensor fields for which the two factors T_{i_1}, T_{i_2} in $C^{l,i_1 i_2}_g$ have three, two, one and no particular contractions between them, respectively. Now, given any $C^{l,i_1 i_2}_g$, $l \in L_{2,diff}$, we denote by $\hat{C}^l_g$ the complete contraction that arises by contracting ${}_{i_1}$ against ${}_{i_2}$ and then hitting the factors T_{i_1}, T_{i_2} by derivatives $\nabla_{i_*}, \nabla^{i_*}$ (which contract against each other).*

Now, given a complete contraction C^t_g, $t \in T$, let us denote by $Pair^\alpha_t$, $Pair^\beta_t$, $Pair^\gamma_t$, $Pair^\delta_t$ the set of (ordered) pairs of factors (T_a, T_b) in C^t_g that have four, three, two and one particular contractions between them, respectively. Given any $C^t_g, t \in T$ and any pair of factors $(T_a, T_b) \in Pair^\alpha_t$ etc, we denote by $Hit_{T_a,T_b}[C^t_g]$ the complete contraction in $Xdiv_{i_1}\nabla_{i_1} C^t_g$ where the derivative ∇_{i_1} is forced to hit T_a and then the derivative ∇^{i_1} in $Xdiv_{i_1}$ is forced to hit T_b.

[181](Using the last Lemma in the Appendix of [3]).

We will then prove that:

$$\sum_{l\in L^{\alpha}_{2,diff}} a_l \hat{C}^l_g = \sum_{t\in T} a_t \sum_{(T_a,T_b)\in Pair^{\alpha}_t} Hit_{T_a,T_b}[C^t_g], \tag{9.59}$$

$$\sum_{l\in L^{\beta}_{2,diff}} a_l \hat{C}^l_g = \sum_{t\in T} a_t \sum_{(T_a,T_b)\in Pair^{\beta}_t} Hit_{T_a,T_b}[C^t_g], \tag{9.60}$$

$$\sum_{l\in L^{\gamma}_{2,diff}} a_l \hat{C}^l_g = \sum_{t\in T} a_t \sum_{(T_a,T_b)\in Pair^{\gamma}_t} Hit_{T_a,T_b}[C^t_g], \tag{9.61}$$

$$\sum_{l\in L^{\delta}_{2,diff}} a_l \hat{C}^l_g = \sum_{t\in T} a_t \sum_{(T_a,T_b)\in Pair^{\delta}_t} Hit_{T_a,T_b}[C^t_g]. \tag{9.62}$$

We note that if we can prove the above, then by just applying the eraser to the pair of contracting derivative indices $\nabla^{i_*}, \nabla_{i_*}$[182] in all four of the above equations and then adding, we derive (9.56).

Let us first prove (9.59), which contains (in a simple form) the idea for the proof of the other equations. We will then prove (9.60), and explain how the other two equations follow by the same argument.

Proof of (9.59): Pick out the sublinear combination in (1.6) with two factors $\nabla_a R_{ijkl}, \nabla_{a'} R_{i'j'k'l'}$,[183] with *five* particular contractions between them. It is clear that this sublinear combination must vanish separately, and that the only terms in $\sum_{t\in T} a_t X div_{i_1} C^{t,i_1}_g$ that can contribute to this sublinear combination from are precisely the terms $\sum_{t\in T} a_t \sum_{(T_a,T_b)\in Pair^{\alpha}_t} Hit_{T_a,T_b}[C^t_g]$. From $\sum_{l\in L_{2,diff}} a_l X div_{i_1} X div_{i_2} C^{l,i_1 i_2}_g$, the only terms that can contribute to this are the tensor fields indexed in $L^{\alpha}_{2,diff}$, if we *force* the derivative ∇^{i_1} to hit T_{i_2} and ∇^{i_2} to hit T_{i_1}. Our claim then just follows by the second Bianchi identity. $\Box$

Proof of (9.60), (9.61), (9.62): Pick out the sublinear combination in (1.6) with two factors $\nabla_a R_{ijkl}, \nabla_{a'} R_{i'j'k'l'}$ with *four* particular contractions between them. Denote this sublinear combination (which clearly must vanish separately) by $Y_g = 0$. Let us also observe (by virtue of the Bianchi identities and the antisymmetries of the curvature tensor) that the four particular contractions can either be according to the pattern $\nabla_a R_{ijkl} \nabla_{a'} R^{ijkl}$, *or* according to the pattern $\nabla_a R_{ijkl} \nabla^a R_{i'}{}^{jkl}$; denote these two sublinear combination by Y^1_g, Y^2_g. It follows (using the variation of the curvature tensor by a symmetric 2-tensor v_{ij}), that $Y^1_g = 0, Y^2_g = 0$.

Now, notice that the only terms in $\sum_{t\in T} a_t X div_{i_1} C^{t,i_1}_g$ that can contribute to Y^2_g are precisely the terms $\sum_{t\in T} a_t \sum_{(T_a,T_b)\in Pair^{\alpha}_t} Hit_{T_a,T_b}[C^t_g]$.
From $\sum_{l\in L_{2,diff}} a_l X div_{i_1} X div_{i_2} C^{l,i_1 i_2}_g$, the only terms that can contribute to

[182]Note that by weight considerations there is exactly one such pair.
[183]We are again not writing out the indices in those factors that contract against factors $\nabla\phi_h$).

this are the tensor fields indexed in $L^{\alpha}_{2,diff}$, if we *force* the derivative ∇^{i_1} to hit T_{i_2} and ∇^{i_2} to hit T_{i_1}, and then apply the second Bianchi identity to replace any expression $\nabla^{i'}R_{ijkl}\nabla^i R^{i'}{}_{j'}{}^{k'l'}$ by $\nabla^s R_{ijkl}\nabla_s R^i{}_j{}^{kl}$ (the *other term* that arises in the second Bianchi identity falls into Y^1_g), or $\nabla^{i'}R_{ijkl}\nabla^i R_{i'}{}^{jk}{}_{l'}$ (no correction term in this case).

(9.61), (9.62) follow by the same reasoning. Thus we derive our claim. □

We have shown Lemma 1.3 when $\mu = 1$, except for the special cases described at the beginning of this subsection. We derive Lemma 1.3 in those special settings in the Mini-Appendix below:

9.8 Mini-Appendix: Proof of Lemma 1.3 when $\mu = 1$, in the "special subcases".

We now prove our claim in the special subcases, defined at the beginning of subsection (9.7); we recall that $\sigma_2 > 0$ in the special subcases. We recall that by virtue of the assumption $L^+_\mu = \emptyset$ in our Lemma assumption, we may assume wlog that all tensor fields of minimum rank 1 in (1.6) contain no free index in any factor of the form $S_*\nabla^{(\nu)}R_{ijkl}$. We first consider the special subcases where the ‘-tensor fields of maximal refined double character in (1.6) correspond to (9.43):

The second special case, (9.43): We recall also that no 2-tensor field in (1.6) can contain more than one free index in any simple factor $S_*\nabla^{(\nu)}R_{ijkl}$, by weight considerations. Again by weight considerations, we derive that the maximum rank of tensor fields appearing in (1.6) is 2, and moreover those tensor fields can have no removable free index. (In particular any given simple factor $S_*\nabla^{(\nu)}R_{ijkl}$ must have $\nu = 0$). Moreover, for any μ-tensor field in (1.6) and any given factor $S_*\nabla^{(\nu)}R_{ijkl}\nabla^i\tilde{\phi}_h$ can have $\nu = 0$ or $\nu = 1$.

Now, denote by $L^x_1 \subset L_1$ the index set of 1-tensor fields in (1.6) with $\nu = 0$ on the factor $S_*\nabla^{(\nu)}R_{ijkl}\nabla^i\tilde{\phi}_1$. We denote by $L^q_1 \subset L_1$ the index set of μ-tensor fields in (1.6) with $\nu = 1$ in the factor $S_*\nabla^{(\nu)}R_{ijkl}\nabla^i\tilde{\phi}_1$. By our remark above, $L^x_1 \bigcup L^q_1 = L_1$.

We will prove the following: First, that there exists a linear combination of 2-tensor fields (indexed in H' below), as required by Lemma 1.3, such that:

$$\begin{aligned}&\sum_{l\in L^x_1} a_l C^{l,i_1}_g \nabla_{i_1}\upsilon - Xdiv_{i_2}\sum_{h\in H'} a_h C^{h,i_1i_2}_g \nabla_{i_1}\upsilon = \\ &\sum_{l\in L'_1} a_l C^{l,i_1}_g \nabla_{i_1}\upsilon + \sum_{j\in J} a_j C^{j,i_1}_g \nabla_{i_1}\upsilon.\end{aligned} \tag{9.63}$$

Here $\sum_{l\in L'_1} a_l C^{l,i_1}_g$ stands for a linear combination of acceptable μ-tensor fields in the form (1.5) with a u-simple character $\vec{\kappa}_{simp}$, and in addition have $\nu = 1$ in the factor $S_*\nabla^{(\nu)}R_{ijkl}\nabla^i\tilde{\phi}_1$. The terms indexed in J are simply subsequent to $\vec{\kappa}_{simp}$. The above holds modulo terms of length $\geq \sigma + u + 2$. Thus, making the $\nabla\upsilon$ into an $Xdiv$ and replacing into (1.6), we may assume wlog that $L^x_\mu = \emptyset$.

We then will prove that:

$$\sum_{h\in H'\bigcup L_{>1}} a_h C^{h,i_1 i_2}_g \nabla_{i_1}\upsilon\nabla_{i_2}\upsilon = 0. \qquad (9.64)$$

(Here the terms indexed in $L_{>1}$ are *the same* terms appearing in (1.6); the terms indexed in H' are the same that appear in (9.63). Thus, making the $\nabla\upsilon$ into an Xdi and replacing into (1.6), we may additionaly assume wlog that $H\bigcup H'=\emptyset$. Finally, under that additional assumption we will show that:

$$\sum_{l\in L^x_1} a_l C^{l,i_1}\nabla_{i_1}\upsilon = 0. \qquad (9.65)$$

Clearly, if we can show the above three equations, then our claim will follow.

The proof of the above equation just relies on the trick that we introduced to prove (in [7]) the Lemma 5.1 in [6]:

Pick out the sublinear combination in (1.6) with an (undifferentiated) factor $S_*R_{ijkl}\nabla^i\tilde{\phi}_1$. This sublinear combination must vanish separately, thus we derive:

$$\sum_{l\in L^x_1} a_l X_*div_{i_1}C^{l,i_1} - X_*div_{i_1}X_*div_{i_2}\sum_{h\in H} a_h C^{h,i_1 i_2}_g + \sum_{j\in J} a_j C^j_g = 0. \qquad (9.66)$$

Now, we denote by $H_a\subset L_{>1}$ the index set of 2-tensor fields in the above that contain a free index, say the index ${}_{i_2}$ wlog, in the factor $S_*R_{ijkl}\nabla^i\tilde{\phi}_1$. We also formally replace the factor S_*R_{ijkl} by an expression $\nabla_j\omega\nabla_k\omega\nabla_l y-\nabla_j\omega\nabla_l\omega\nabla_k y$, thus deriving a new true equation. Denote the resulting tensor fields by $\tilde{C}^{l,i_1}, \tilde{C}^{h,i_1 i_2}_g$. We then apply Lemma 5.1 in [6] to the resulting equation. We derive that:

$$\sum_{l\in L^x_1} a_l \tilde{C}^{l,i_1}\nabla_{i_1}\upsilon + X_*div_{i_2}\sum_{h\in H_a} a_h \tilde{C}^{h,i_1 i_2}_g\nabla_{i_1}\upsilon = X_*div_{i_2}\sum_{h\in H'} a_h C^{h,i_1 i_2}_g\nabla_{i_1}\upsilon$$
$$+\sum_{j\in J} a_j C^j_g. \qquad (9.67)$$

Here the terms indexed in H' are of the same form as the terms in the LHS, but have rank 2; furthermore, the free indices ${}_{i_1}, {}_{i_2}$ *do not* belong to any of the factors $\nabla\omega, \nabla y$. Now, by just formally replacing the expression $\nabla_a\omega\nabla_b\omega\nabla_c y$ by $S_*R_{i(ab)c}\nabla^i\tilde{\phi}_1$ (this gives a new true equation), we derive (9.63). (9.64) follows by the exact same argument. (9.65) follows by the same argument, the only difference being that we now pick out the terms with a factor $\nabla_d S_*R_{ijkl}\nabla^i\tilde{\phi}_1$ and replace them by an expression $\nabla_d\omega\nabla_j\omega\nabla_k\omega\nabla_l y-\nabla_d\omega\nabla_j\omega\nabla_l\omega\nabla_k y$. We then apply the Lemma 5.1 in [6],[184], and in the end formally replace each expression

[184] See the Appendix of [6]

$\nabla_d\omega\nabla_j\omega\nabla_k\omega\nabla_l y$ by a factor $\nabla_{(d}S_*R^i{}_{jk)l}\nabla_i\tilde{\phi}_1$. This proves (9.65). □

We now consider the special cases where the 1-tensor fields of maximal refined double character in (1.6) have no removable free indices:

The first special case:

We start by observing that (by weight considerations) there can be no tensor fields of rank higher than 1 in (1.6), and that $\sigma_2 > 0$.

In this case, the claim of Lemma 1.3 is a straightforward application of the "generalized form" of Lemma 5.1 in [6]: Consider a factor $S_*R_{ijkl}\nabla^i\tilde{\phi}_x$ in $\vec{\kappa}_{simp}$. Wlog we asume that $x = 1$. By virtue of the assumption $L^+_\mu = \emptyset$, we know that this factor does not contain a free index for any of the tensor fields in (1.6). We pick out the sublinear combination of terms with an (undifferentiated) factor $S_*R_{ijkl}\nabla^i\tilde{\phi}_1$ in (1.6). This sublinear combination must clearly vanish separately, thus we derive that:

$$\begin{aligned}&\sum_{l\in L_\mu} a_l X_*div_{i_1}\dots X_*div_{i_\mu}C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\\&+\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u) = 0.\end{aligned} \tag{9.68}$$

(Here $X_*div_{i_s}[\dots]$ stands for the sublinear combination in $Xdiv_{i_s}[\dots]$ where the derivative ∇^{i_s} is not allowed to hit the factor $S_*R_{ijkl}\nabla^i\tilde{\phi}_1$; notice that none of the free indices belong to $S_*R_{ijkl}\nabla^i\tilde{\phi}_1$).

Now, let $\tilde{C}^{l,i_1\dots i_\mu}_g$ stand for the tensor field that formally arises from each $C^{l,i_1\dots i_\mu}_g$ by replacing the factor $S_*R_{ijkl}\nabla^i\tilde{\phi}_1$ by an expression $\nabla_j\omega\nabla_k\omega\nabla_l y - \nabla_j\omega\nabla_l\omega\nabla_j y$. We perform this formal replacement to the terms in (9.68), and then we apply Lemma 5.1 in [6] to the resulting equation. We derive that:

$$\sum_{l\in L_\mu} a_l\tilde{C}^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon = 0.$$

Thus, formally replacing each expression $\nabla_j\omega\nabla_k\omega\nabla_l y$ by $S_*R_{i(jk)l}\nabla^i\tilde{\phi}_1$, we derive our claim in this remaining special case. □

10 Appendix: Proof of Lemmas 3.3, 3.4 in [6].

Definitions related to Lemmas 3.3, 3.4 in [6]: In case A of Lemma 1.3 we denote by $L^*_\mu \subset L_\mu$ the index set of those tensor fields $C^{l,i_1\dots i_\mu}_g$ in (1.6) for which some factor $\nabla^{(A)}_{r_1\dots r_A}\Omega_x$ (for a single x, which we are free to define) has $A = 2$ and both indices $_{r_1}, _{r_2}$ are free indices.

Also, we define $L^+_\mu \subset L_\mu$ to stand for the index set of those μ-tensor fields that have a free index ($_{i_\mu}$ say) belonging to a factor $S_*R_{ijkl}\nabla^i\tilde{\phi}_h$ (without derivatives).

We now consider the sublinear combination indexed in $L_{>\mu}$ in (1.6). We define $L''_+ \subset L_{>\mu}$ to stand for the index set of $(\mu+1)$-tensor fields with a factor $S_*R_{ijkl}\nabla^i\tilde{\phi}_h$ with *both* indices ${}_j, {}_k$ free.

The Lemmas proven in the present paper: After all these definitions, we are prepared to re-state Lemmas 3.3, 3.4 in [6]:

Lemma 10.1 *Assume (1.6), where the terms in the LHS of that equation have weigh $-n$, real length σ, Φ factors $\nabla\phi, \nabla\phi', \nabla\tilde{\phi}$ and $\sigma_1+\sigma_2$ curvature factors $\nabla^{(m)}R_{ijkl}, S_*\nabla^{(\nu)}R_{ijkl}$;*[185] *assume also that no μ-tensor field there has any special free indices. We claim that there is a linear combination of acceptable $(\mu+1)$-tensor fields, $\sum_{p\in P} a_p C^{p,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ with a simple character $\vec{\kappa}_{simp}$ so that:*

$$\begin{aligned}
&\sum_{l\in L^*_\mu\bigcup L^+_\mu} a_l C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon+\\
&\sum_{p\in P} a_p Xdiv_{i_{\mu+1}} C^{p,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon=\\
&\sum_{j\in J} a_j C^{j,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon+\\
&\sum_{l\in\tilde{L}} a_l C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon,
\end{aligned} \tag{10.1}$$

modulo complete contractions of length $\geq \sigma+u+\mu+1$. The tensor fields indexed in J on the right hand side are simply subsequent to $\vec{\kappa}_{simp}$. The terms indexed in $\tilde{L}'$ in the RHS are acceptable terms in the form (1.5) with a simple character $\vec{\kappa}_{simp}$. They are not in the forms corresponding to tensor fields indexed in $L^*_\mu\bigcup L^+_\mu$.

Lemma 10.2 *Assume (1.6) with weight $-n$, real length σ, Φ factors $\nabla\phi, \nabla\phi', \nabla\tilde{\phi}$ and $\sigma_1+\sigma_2$ factors $\nabla^{(m)}R_{ijkl}, S_*\nabla^{(\nu)}R_{ijkl}$; assume also that none of the μ-tensor fields have special free indices, and that $L^*_\mu\bigcup L^+_\mu=\emptyset$. We claim that there exists a linear combination of acceptable $(\mu+2)$-tensor fields, $\sum_{p\in P} a_p C^{p,i_1\dots i_{\mu+2}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ with simple character $\vec{\kappa}_{simp}$, so that:*

[185] See the discussion on the *induction* in the introduction.

$$
\begin{aligned}
&\sum_{l\in L''_+} a_l Xdiv_{i_1}\dots Xdiv_{i_{\mu+1}} C^{l,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)+\\
&\sum_{p\in P} a_p Xdiv_{i_1}\dots Xdiv_{i_{\mu+2}} C^{p,i_1\dots i_{\mu+2}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)=\\
&\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\\
&+\sum_{l\in\tilde{L}'} a_l Xdiv_{i_1}\dots Xdiv_{i_{\mu+1}} C^{l,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u),
\end{aligned}
\tag{10.2}
$$

modulo complete contractions of length $\ge \sigma+u+1$. $\sum_{j\in J}\dots$ stands for a linear combination of complete contractions that are simply subsequent to $\vec{\kappa}_{simp}$. The terms indexed in $\tilde{L}'$ in the RHS are acceptable terms in the form (1.5) with a simple character $\vec{\kappa}_{simp}$. They are not in the forms corresponding to tensor fields indexed in L''_+.

Proof of Lemma 10.1: We prove Lemma 10.1 by breaking it into two Lemmas. In Lemma 10.3 below, we aim to "get rid" of the tensor fields indexed in L^*_μ. (Recall that $L^*_\mu \subset L_\mu$ stands for the index set of factors with two free indices belonging to a factor $\nabla^{(2)}\Omega_x$.

With no loss of generality (up to re-labelling the functions Ω_h) we may assume that $x=1$.

Lemma 10.3 *We claim that there is a linear combination of acceptable $(\mu+1)$-tensor fields, $\sum_{p\in P} a_p C^{p,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ so that:*

$$
\begin{aligned}
&\sum_{l\in L^*_\mu} a_l C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon+\\
&\sum_{p\in P} a_p Xdiv_{i_{\mu+1}} C^{p,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon=\\
&\sum_{j\in J} a_j C^{j,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon\\
&+\sum_{l\in\tilde{L}} a_l C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon.
\end{aligned}
\tag{10.3}
$$

where each $C^{l,i_1\dots i_\mu}_g$ on the RHS is acceptable in the form (1.5) with a u-simple character $\vec{\kappa}_{simp}$ has a factor $\nabla^{(A)}\Omega_1$ with $A\ge 3$. The complete contractions indexed in J are simply subsequent to $\vec{\kappa}_{simp}$. The above holds modulo complete contractions of length $\ge \sigma+u+\mu+1$.

If we can show the above, then we will be reduced to showing Lemma 10.1 under the assumption that $L^*_\mu=\emptyset$.

Our next claim, which will "get rid" of the sublinear combination indexed in L^+_μ: In fact, we make a stronger claim:

Lemma 10.4 *Assume (1.6) where no μ-tensor field there has special free indices;*[186] *consider a given simple factor $S_*\nabla^{(\nu)}R_{ijkl}$.*[187] *Let $\tilde{L}^+_\mu \subset L^+_\mu$ stand for the index set of tensor fields which contain exactly one free index in the selected factor $S_*\nabla^{(\nu)}R_{ijkl}$. We claim that there is a linear combination of acceptable $(\mu+1)$-tensor fields, $\sum_{p\in P} a_p C^{p,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ so that:*

$$\begin{aligned}
&\sum_{l\in L^+_\mu} a_l C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon+\\
&\sum_{p\in P} a_p Xdiv_{i_{\mu+1}} C^{p,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon=\\
&\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u,\upsilon^\mu).
\end{aligned} \tag{10.4}$$

Clearly, if we can show the two Lemmas above, then Lemma 10.1 will follow. Now, Lemma 10.2 will also follow by two claims. Recall the definitions of the sets L''_+ from the previous subsection. We then claim:

Lemma 10.5 *In the notation above, and under the assumption that $L^+_\mu=\emptyset$, we claim that there exists an acceptable linear combination of $(\mu+2)$-tensor fields, $\sum_{p\in P} a_p C^{p,i_1\dots i_{\mu+2}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)$ so that:*

$$\begin{aligned}
&\sum_{l\in L''_+} a_l Xdiv_{i_1}\dots Xdiv_{i_{\mu+1}} C^{l,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)+\\
&\sum_{p\in P} a_p Xdiv_{i_1}\dots Xdiv_{i_{\mu+2}} C^{p,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)=\\
&\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\\
&+\sum_{l\in\tilde{L}'} a_l Xdiv_{i_1}\dots Xdiv_{i_{\mu+1}} C^{l,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u),
\end{aligned} \tag{10.5}$$

modulo complete contractions of length $\ge \sigma+u+1$. Here each $C^{l,i_1\dots i_{\mu+1}}$ in the RHS is acceptable in the form (1.5), with a simple character $\vec{\kappa}_{simp}$, and has no factors S_*R_{ijkl} with two free indices.

Clearly, if we can show Lemma 10.5, then Lemma 10.2 will follow.

10.1 Proof of Lemma 10.3.

Observe that by hypothesis $\mu\ge 2$ in (1.6) in this setting. We now prove our claim in all cases except for a "special subcase", which will be treated at the end

[186] Recall that a free index is called "special" if it occupies the position $_{k,l}$ in a factor $S_*\nabla^{(\nu)}R_{ijkl}$ or the position $_{i,j,k,l}$ in a factor $\nabla^{(m)}R_{ijkl}$.

[187] Recall that a factor $S_*\nabla^{(\nu)}R_{ijkl}$ is called "simple" when it is contracting against no factor $\nabla\phi'_h$ in the simple character $\vec{\kappa}_{simp}$.

of this subsection. The "special subcase" is when $\mu = 2$ and the terms indexed in L^*_μ have the two free indices in a factor $\nabla^{(2)}\Omega_h$, *and all other factors have no removable free indices.*[188] We now prove our claim, assming that (1.6) does not fall under the "special subcase".

Wlog, by just re-naming factors, let us assume that $h = 1$. (So the two free indices belong to the factor $\nabla^{(2)}\Omega_1$).

We then let $\sum_{q\in Q} a_q C^q_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u, \phi_{u+1})$ stand for a generic linear combination of complete contractions (not necessarily acceptable) with length $\sigma + u + 1$ and a factor $\nabla\phi_{u+1}$, contracting against a factor $\nabla^{(A)}\Omega_1$ with $A \geq 2$.

We also denote by $\sum_{c\in C} a_c C^{c,i_1\dots i_a}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u, \phi_{u+1})$ a generic linear combination of tensor fields with a u-simple character $\vec{\kappa}_{simp}$, with precisely one un-acceptable factor $\nabla\Omega_x$, contracting against a factor $\nabla\phi_{u+1}$ and $a \geq \mu$. Finally, we let $\sum_{j\in J} a_j C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u, \phi_{u+1})$ stand for a linear combination of contractions with length $\sigma + u + 1$ and simply subsequent to $\vec{\kappa}_{simp}$.

We make $F_1 = \nabla^{(A)}\Omega_1$ the selected factor and consider the equation $Image^{1,+}_{\phi_{u+1}}[L_g] = 0$, see equation (6.26). *Note: By the discussion regarding the derivation of the super divergece formula above, this equation can always be applied to the equation $L_g = 0$–we do not need any assumptions regarding the sublinear combinations indexed in L^*_μ, L^+_μ, etc.* By our choice of selected factor, all contractions in $Image^{1,+}_{\phi_{u+1}}[L_g]$ will contain at least $\sigma + u + 1$ factors. For each $l \in L^*_\mu$ we denote by $F_2, \dots F_\sigma$ the real factors other than $F_1 = \nabla^{(A)}\Omega_1$. We thus derive:

$$\begin{aligned}
&\sum_{l\in L^*_\mu} a_l Xdiv_{i_3} \dots Xdiv_{i_\mu} Xdiv_{i_*} \sum_{k=2}^{\sigma} C^{l,i_3\dots i_\mu,k(i_*)}_g(\Omega_2, \dots, \Omega_p, \phi_1, \dots, \phi_u) \\
&\nabla^{i_1}\Omega_1 \nabla_{i_1}\phi_{u+1} + \sum_{c\in C} a_c Xdiv_{i_1} \dots Xdiv_{i_a} C^{c,i_1\dots i_a}_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u, \phi_{u+1}) \\
&+ \sum_{q\in Q} a_q C^q_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u, \phi_{u+1}) \\
&+ \sum_{j\in J} a_j C^j_g(\Omega_1, \dots, \Omega_p, \phi_1, \dots, \phi_u, \phi_{u+1}) = 0,
\end{aligned} \tag{10.6}$$

modulo complete contractions of length $\geq \sigma + u + 2$. Here each tensor field $C^{l,i_3\dots i_\mu,k(i_*)}_g$ arises from $C^{l,i_1\dots i_\mu}_g$ by erasing the factor $F_1 = \nabla^{(2)}_{i_1 i_2}\Omega_1$ and hitting the k^{th} factor F_k by a derivative index ∇_{i_*}. In the above equation we pick out the sublinear combination of contractions containing an expression $\nabla_i\Omega_1\nabla^i\phi_{u+1}$. This sublinear combination must vanish separately, since (10.6)

[188] See definition 4.1 in [6]. Notice that by weight considerations, if this is true of one term in L^*_μ then it will be true of all of them.

holds formally. Denote the resulting true equation by $S_g = 0$. Now, we formally erase the expression $\nabla_i \Omega_1 \nabla^i \phi_{u+1}$ in S_g, and we obtain a new true equation:

$$\sum_{l \in L^*_\mu} a_l Xdiv_{i_3} \dots Xdiv_{i_\mu} Xdiv_{i_*} \sum_{k=1}^{\sigma-1} C^{l,i_1 \dots i_\mu, k(i_*)}_g (\Omega_2, \dots, \Omega_p, \phi_1, \dots, \phi_u)$$
$$+ \sum_{c \in C} a_c Xdiv_{i_1} \dots Xdiv_{i_a} C^{c,i_1 \dots i_a}_g (\Omega_2, \dots, \Omega_p, \phi_1, \dots, \phi_u) +$$
$$\sum_{j \in J} a_j C^j_g (\Omega_2, \dots, \Omega_p, \phi_1, \dots, \phi_u) = 0, \tag{10.7}$$

which holds modulo complete contractions of length $\geq \sigma + u$. Here $C^{c \dots}$ has arisen from the previous equation by just erasing the expression $\nabla_i \Omega_1 \nabla^i \phi_{u+1}$. We denote by $\vec{\kappa}'_{simp}$ the $(u-1)$-simple character of those complete contractions. The contractions C^j in this setting have at least one factor $\nabla \phi_h, h \in Def(\vec{\kappa}_{simp})$ contracting against a derivative index.

Now, we prove Lemma 10.3 in pieces. Consider the μ-tensor fields in L^*_μ. We subdivide L^*_μ into subsets $L^{*,z}_\mu, z \in Z$ so that two tensor fields $C^{l_1,i_1 \dots i_\mu}, C^{l_2,i_1 \dots i_\mu}$ with the same refined double character $\vec{\kappa}^z_{ref-doub}$ will be indexed in the same index set $L^{*,z}_\mu$. We let $Z_{Max} \subset Z$ stand for the index sets corresponding to the maximal refined double characters. Suppose $M (\geq 1)$ is the maximum number of free indices that can appear in a given factor $F_d \neq F_1$ among all the tensor fields indexed in L^*_μ. Then, by the definition on the maximal refined double character in [6], M will also be the maximum number of free indices that can appear in a given factor among all the tensor fields indexed in $\bigcup_{z \in Z_{Max}} L^{*,z}_\mu$. We denote by $cut(\vec{\kappa})^z_{ref-doub}$ the refined double character that formally arises from $\vec{\kappa}^z_{ref-doub}$ by erasing the entry that corresponds to $\nabla^{(2)}_{i_1 i_2} \Omega_1$. Now, among all the factors $F_d, d \neq 1$ in all the tensor fields indexed in $\bigcup_{z \in Z_{Max}} L^{*,z}_\mu$ that have M free indices, we pick out one (or a category of generic factors $\nabla^{(m)} R_{ijkl}$) canonically, using the same method that was used to choose the critical factor in [6]. We call that (set of) factors the (set of) α-factor(s). We index in $Z'_{Max} \subset Z_{Max}$ the set of maximal refined double characters $\vec{\kappa}^z_{ref-doub}, z \in Z_{Max}$ that have M free indices in the (an) α-factor. Now, for each $z \in Z'_{Max}$, we denote by $\tilde{\kappa}^z_{ref-doub}$ the refined double character that arises from $cut(\vec{L})^{*,z}_\mu$ by formally adding a (derivative) free index ∇_{i_*} onto the (one of the) α-factor(s).

We consider (10.7) and we observe that the maximal refined double characters among the tensor fields in Erase[(10.7)] will be $\vec{\kappa}^z_{ref-doub}$. Now, assume with no loss of generality that the α-factor(s) is (are) F_2 $(F_2, \dots, F_d)$. Then, in (10.7) the $(\mu - 1)$-tensor fields with $(M+1)$ free indices on the (one of the) α-factor(s) will be precisely the sublinear combination in the first line with $k = 2$ (or $k = 2, \dots, d$). Therefore, applying our inductive assumption of Proposition

1.1,[189] we derive that for each $z \in Z'_{Max}$ there is a linear combination of acceptable μ-tensor fields with a $(\mu-1)$-refined double character $\tilde{L}^{*,z}_\mu$, indexed in P below so that:

$$\begin{aligned}
&\sum_{l \in L^*_\mu} a_l \sum_{k=1}^{d} C^{l,i_1\dots i_\mu,k(i_*)}_g(\Omega_2,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_3}\upsilon\dots\nabla_{i_*}\upsilon+\\
&\sum_{p\in P} a_p X div_{i_\mu} C^{p,i_1\dots i_{\mu-1}i_\mu}_g(\Omega_2,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_{\mu-1}}\upsilon = \\
&\sum_{j\in J} a_j C^j_g(\Omega_2,\dots,\Omega_p,\phi_1,\dots,\phi_u,\upsilon^{\mu-1}).
\end{aligned} \tag{10.8}$$

Now, in the case $\mu > 2$, we prove our claim by our standard formal manipulation of the above equation: In that case, we identify in each of the contractions above the one factor that is contracting against the most factors $\nabla\upsilon$-it will be the α-factor which will be contracting against $M+1 \geq 2$ factors $\nabla\upsilon$. Then, we erase one of the factors $\nabla\upsilon$ that is contracting against the α-factor (thus obtaining a new true equation) and we multiply the new true equation by $\nabla_{ij}\Omega_1\nabla^i\upsilon\nabla^j\upsilon)$.[190] This further true equation is precisely the claim of Lemma 10.3 (when $\mu > 2$). In the case, $\mu = 2$, we first apply Lemma 4.6 or Corollary 2 or Corollary 3 from [6],[191] to ensure that for the terms indexed in P above, the (unique) factor $\nabla\upsilon$ contracts against a derivative index, and if it contracts against a factor $\nabla^{(B)}\Omega_x$ then $B \geq 3$. With this extra restriction, we repeat the argument above and derive our claim. □

Proof of Lemma 10.3 in the "special subcases": Let us write out:

$$\sum_{l\in L^*_\mu} a_l C^{l,i_1 i_2}_g = \sum_{y\in Y} a_y C^y_g \cdot \nabla^{(2)}_{i_1 i_2}\Omega_1.$$

(In other words, we "factor out" the term $\nabla^{(2)}_{i_1 i_2}\Omega_1$ which contains the two free indices); we are then left with a complete contraction.

Now, we apply the "inverse integration by parts" technique which was introduced in section 3 in [7], and then apply the "silly divergence formula", obtained by integrating by parts with respect to the function Ω_1.[192] Pick out the sublinear combination of terms with length $\sigma + u$, with *no* internal contractions and with u factors $\nabla\phi_h$. The resulting equation will be in the form:

---

[189] Since all the tensor fields of minimum rank in (10.7) have all free indices being non-special, there is no danger of falling under a "forbidden case" of that Proposition.

[190] Since $\mu > 2$ it follows that $M + 1 > 1$; thus since all factors $\nabla\upsilon$ contract against non-special indices, at least one of the $M+1$ factors $\nabla\upsilon$ will be a derivative index, hence the Eraser can be applied.

[191] The fact that we are not dealing with the "special subcase" ensures that (10.8) does not fall under a "forbidden case" of any of those Lemmas, by weight considerations.

[192] Refer to section 3 in [7] for a detailed description of these operations.

$$\Omega_1 \cdot \sum_{y\in Y} a_y Spread^{\nabla^t,\nabla_t}\{Spread^{\nabla^s,\nabla_s}[C^t_g]\} = 0.$$

(Recall that $Spread^{\nabla^t,\nabla_t}$ stands for a formal operation that acts on complete contractions in the form (1.5) by hitting two different factors by derivatives ∇^t,∇_t that contract against each other, and then adding over all the terms we can thus obtain).

Now, since the above holds formally, we derive that:

$$\sum_{y\in Y} a_y C^t_g = 0.$$

Multiplying the above by $\nabla^{(2)}_{i_1 i_2}\Omega_1\nabla^{i_1}\upsilon\nabla^{i_2}\upsilon$, we derive our claim in the special subcase. □

10.2 Proof of Lemmas 10.4 and 10.5.

Notation: Firstly, we denote by $L^\nu \subset L_\mu$ the index set of the μ-tensor fields indexed in L_μ, for which the special factor $S_*\nabla^{(r)}R_{ijkl}$ has the index $_i$ contracting against a factor $\nabla^i\tilde{\phi}_\nu$ and contains exactly one (non-special) free index. We will also denote by $\overline{L}^\nu$ the index set of the $(\mu+1)$-tensor fields in (1.6) which have two free indices in the expression $S_*\nabla^{(\rho)}R_{ijkl}\nabla^i\tilde{\phi}_\nu$, one of which is special–we assume wlog that the special free index is $_k = {}_{i_{\mu+1}}$.

Lemma 10.6 *In the notation above, we claim that there exists a linear combination of $(\mu+1)$-tensor fields, with a u-simple character $\vec{\kappa}_{simp}$ and with certain additional properties explained below (10.9) so that:*

$$\begin{aligned}
&\sum_{l\in L^\nu} a_l C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon + \\
&\sum_{l\in \overline{L}^\nu} a_l Xdiv_{i_{\mu+1}} C^{l,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon - \\
&Xdiv_{i_{\mu+1}}\sum_{p\in P} a_p C^{p,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon = \\
&\sum_{j\in J} a_j C^{j,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon.
\end{aligned} \tag{10.9}$$

The additional properties of the tensor fields indexed in P are as follows: Firstly only the index $_{i_\mu}$ among the above free indices $_{i_1},\dots,_{i_\mu}$ belongs to the (special) factor $S_*\nabla^{(\rho)}R_{ijkl}\nabla^i\tilde{\phi}_\nu$, and secondly if $_{i_{\mu+1}}$ does belong to the above factor then $\rho > 0$.

We observe that if we can prove the above then by making the $\nabla\upsilon$s into $Xdiv$s (by virtue of the last Lemma in the Appendix of [3]) we can derive both

Lemma 10.4, and Lemma 10.5 in case B.

Proof of Lemma 10.6:

Definition 10.1 *We denote by $Cut(\vec{\kappa}_{simp})$ the $(u-1)$-simple character that formally arises from $\vec{\kappa}_{simp}$ by replacing the expression $S_*\nabla^{(\rho)}R_{ijkl}\nabla^i\tilde{\phi}_\nu$ by a factor $\nabla^{(\rho+2)}Y$ (Y is treated as a function Ω_{p+1}).*

We then denote by $C^{l,i_1\dots i_\mu|A}_g$ the tensor fields that arises from $C^{l,i_1\dots i_\mu}_g$ by replacing the expression $S_*\nabla^{(\rho)}_{r_1\dots r_\rho}R_{ijkl}\nabla^i\tilde{\phi}_\nu$ by a factor $\nabla^{(\rho+2)}_{r_1\dots r_\rho jk}Y\nabla_l\phi_\nu$. We also denote by $C^{l,i_1\dots i_\mu|B}_g$ the tensor field that arises from $C^{l,i_1\dots i_\mu}_g$ by formally replacing the expession $S_*\nabla^{(\rho)}_{r_1\dots r_\rho}R_{ijkl}\nabla^i\tilde{\phi}_\nu$ by $-\nabla^{(\rho+2)}_{r_1\dots r_\rho jl}Y\nabla_k\phi_\nu$.

Analogously, for each $l\in\overline{L}^\nu$ we denote by $C^{l,i_1\dots i_{\mu+1}|A}_g$ the tensor field that arises from by by replacing the expression $S_*\nabla^{(\rho)}_{r_1\dots r_\rho}R_{ijkl}\nabla^i\tilde{\phi}_\nu$ by $\nabla^{(\rho+2)}_{r_1\dots r_\rho jk}Y\nabla_l\phi_\nu$ ($_l$ is not a free index). We also denote by $C^{l,i_1\dots i_\mu|B}_g$ the tensor field that arises from $C^{l,i_1\dots i_\mu}_g$ by replacing the factor $S_*\nabla^{(\rho)}_{r_1\dots r_\rho}R_{ijkl}\nabla^i\tilde{\phi}_\nu$ by $-\nabla^{(\rho+2)}_{r_1\dots r_\rho jl}Y\nabla_k\phi_\nu$ (now $_k$ is the free index $_{i_{\mu+1}}$).

A note is in order: When we refer to the tensor field $C^{l,i_1\dots i_{\mu+1}|A}_g$ below and we write $X\tilde{div}_{i_{\mu+1}}C^{l,i_1\dots i_{\mu+1}|A}_g$, $X\tilde{div}_{i_{\mu+1}}$ will stand for the sublinear combination in $Xdiv_{i_{\mu+1}}$ where $\nabla^{i_{\mu+1}}$ is *not* allowed to hit the factor $\nabla^{(B)}Y$. Furthermore, when we write $X\tilde{div}_{i_{\mu+1}}C^{l,i_1\dots i_{\mu+1}|B}_g$ below, $X\tilde{div}_{i_{\mu+1}}$ will stand for the regular $Xdiv_{i_{\mu+1}}$ but we will "forget" this structure–i.e. we will treat as a sum of μ-tensor fields.

We will now denote by

$$\sum_{u\in U}a_uC^{u,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u)$$

a generic linear combination of a-tensor fields ($a\ge\mu+1$) with length $\sigma+u$, with the factor $\nabla\phi_\nu$ *not* contracting against the factor $\nabla^{(A)}Y$ and *not* containing a free index.

Now, considering the sublinear combination in $Image^1_Y[L_g](=0)$ which consists of terms where the factor $S_*\nabla^{(\rho)}R_{ijkl}\nabla^i\tilde{\phi}_\nu$, is replaced by $\nabla^{(\rho+2)}Y$ and $\nabla\phi_\nu$ is *not* contracting against $\nabla^{(\rho+2)}Y$, we derive a new true equation:

$$
\begin{aligned}
&\sum_{l\in L^\nu\bigcup(L_\mu\setminus L^\nu)} a_l Xdiv_{i_1}\dots Xdiv_{i_\mu}\{C^{l,i_1\dots i_\mu|A}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u)\\
&+C^{l,i_1\dots i_\mu|B}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u)\}+\\
&\sum_{l\in\overline{L}^\nu} a_l Xdiv_{i_1}\dots Xdiv_{i_\mu}X\tilde{div}_{i_{\mu+1}}\{C^{l,i_1\dots i_{\mu+1}|A}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u)\\
&+C^{l,i_1\dots i_{\mu+1}|B}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u)\}+\\
&\sum_{u\in U} a_u Xdiv_{i_1}\dots Xdiv_{i_a}C^{u,i_1\dots i_a}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u)+\\
&\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u)=0;
\end{aligned}
\tag{10.10}
$$

here the terms indexed in J are simply subsequent to the simple character $Cut(\vec{\kappa}_{simp})$.

We then apply the inductive assumption of Lemma 4.10 in [6][193] to the above, and pick out the sublinear combination of terms where one factor $\nabla\upsilon$ is contracting against the factor $\nabla^{(B)}Y$ and the other $\mu-1$ factors $\nabla\upsilon$ are contracting against other factors. This sublinear combination must vanish separately, thus we derive a new equation:

$$
\begin{aligned}
&\sum_{l\in L^\nu} a_l\{C^{l,i_1\dots i_\mu|A}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u)\\
&+C^{l,i_1\dots i_\mu|B}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u)\}\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon+\\
&\sum_{l\in\overline{L}^\nu_{\mu+1}} a_l X\tilde{div}_{i_{\mu+1}}\{C^{l,i_1\dots i_{\mu+1}|A}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u)\\
&+C^{l,i_1\dots i_{\mu+1}|B}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u)\}\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon+\\
&\sum_{u\in U} a_u Xdiv_{i_{\mu+1}}C^{u,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon+\\
&\sum_{j\in J} a_j C^j_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u,\upsilon^\mu)=0.
\end{aligned}
\tag{10.11}
$$

(Here the tensor fields indexed in U are generic acceptable $(\mu+1)$-tensor fields with a u-simple character $Cut(\vec{\kappa}_{simp})$–the free index $_{i_{\mu+1}}$ *does not* belong to the factor $\nabla\phi_\nu$).

Now, we define an operation $Op[\dots]$ which acts on the tensor fields above by replacing the expression $\nabla^{(B)}_{r_1\dots r_B}Y\nabla^{r_B}\upsilon\nabla_a\phi_\nu$ by an expression $\nabla^{(B+1)}_{r_1\dots r_{B-1}ar_B}\phi_\nu\nabla^{r_B}\upsilon$

[193] Notice that some tensor fields of minimum rank μ in (10.10), i.e. the ones indexed in L^ν, will have only non-special free indices, therefore there is no danger of falling under a "forbidden case" of that Lemma.

(denote the $(u-1)$-simple character that we thus construct by $Cut'(\vec{\kappa}_{simp})$–the factor $\nabla^{(A)}\phi_{u+1}$ is treated as a factor $\nabla^{(A)}\Omega_{p+1}$). Since (10.11) holds formally, we derive:

$$
\begin{aligned}
&\sum_{l\in L^\nu} a_l\{Op[C]^{l,i_1\dots i_\mu|A}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u)\\
&+Op[C]^{l,i_1\dots i_\mu|B}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u)\}\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon+\\
&\sum_{l\in \overline{L}^\nu} a_l X\tilde{div}_{i_{\mu+1}}\{Op[C]^{l,i_1\dots i_{\mu+1}|A}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u)\\
&+Op[C]^{l,i_1\dots i_{\mu+1}|B}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u)\}\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon+\\
&\sum_{u\in U'} a_u Xdiv_{i_{\mu+1}} Op[C]^{u,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon+\\
&\sum_{j\in J} a_j Op[C]^j_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u,\upsilon^\mu)=\\
&\sum_{z\in Z} a_z C^z_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u,\upsilon^\mu);
\end{aligned}
\tag{10.12}
$$

here the tensor fields indexed in Z on the RHS have length $\sigma+u+\mu$ (as opposed to all the terms in the LHS which have length $\sigma+u-1+\mu$), and in addition have a factor $\nabla^{(A)}\phi_{u+1}$ with $A\geq 2$. The correction terms arise by repeating the formal permutations by which the LHS is made formally zero by the LHS of (10.12). The claim $A\geq 2$ follows because the rightmost two indices in each factor will *not* be permuted.

Now, we observe that for each $l\in L^\nu$:

$$
\begin{aligned}
&\{Op[C]^{l,i_1\dots i_\mu|A}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u)+Op[C]^{l,i_1\cdots_\mu|B}_g(\Omega_1,\dots,\Omega_p,\\
&Y,\phi_1,\dots,\phi_u)\}\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon=C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon+\\
&\sum_{z\in Z} a_z C^z_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u,\upsilon^\mu),
\end{aligned}
\tag{10.13}
$$

(where the terms indexed in Z are generic complete contractions as defined above). Analogously, we derive that for each $l\in\overline{L}^\nu$:

$$
\begin{aligned}
&X\tilde{div}_{i_{\mu+1}}\{Op[C]^{l,i_1\dots i_{\mu+1}|A}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u)\nabla_{i_1}\dots\nabla_{i_\mu}\upsilon+\\
&Op[C]^{l,i_1\dots i_{\mu+1}|B}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u)\}\nabla_{i_1}\dots\nabla_{i_\mu}\upsilon=\\
&Xdiv_{i_{\mu+1}}C^{l,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon+\\
&\sum_{z\in Z} a_z C^z_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u,\upsilon^\mu).
\end{aligned}
\tag{10.14}
$$

We then substitute the above two equations into (10.11) and we obtain a new equation:

$$\begin{aligned}
&\sum_{u\in U} a_u Xdiv_{i_{\mu+1}} Op[C]^{u,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon+\\
&\sum_{j\in J} a_j Op[C]^j_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u,\upsilon^\mu)+\\
&\sum_{l\in L^\nu} a_l C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon+\\
&\sum_{l\in \overline{L}^\nu_{\mu+1}} a_l Xdiv_{i_{\mu+1}} C^{l,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon=\\
&\sum_{z\in Z} a_z C^z_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u,\upsilon^\mu),
\end{aligned} \tag{10.15}$$

(denote the $(u-1+\mu)$-simple character o the tensor fields in the LHS of the above by $Ext[\vec{\kappa}_{simp}]$ (the factors $\nabla\upsilon$ are now treated as factors $\nabla\phi_h$).

We now derive our claim from (10.15) via an induction: We inductively assume an equation:

$$\begin{aligned}
&\sum_{u\in U^\delta} a_u Xdiv_{i_{\mu+1}}\dots Xdiv_{i_{\mu+\delta}} \tilde{C}^{u,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_\nu,\phi_1,\dots,\hat{\phi}_\nu,\dots,\phi_u)\nabla_{i_1}\upsilon\\
&\dots\nabla_{i_\mu}\upsilon+\sum_{j\in J} a_j \tilde{C}^j_g(\Omega_1,\dots,\Omega_p,\phi_\nu,\phi_1,\dots,\hat{\phi}_\nu,\dots,\phi_u,\upsilon^\mu)+\\
&\sum_{l\in L^\nu} a_l C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon+\\
&\sum_{l\in \overline{L}^\nu} a_u Xdiv_{i_{\mu+1}} C^{l,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon+\\
&\sum_{p\in P} a_p Xdiv_{i_{\mu+1}} C^{p,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon=\\
&\sum_{z\in Z} a_z C^z_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u,\upsilon^\mu).
\end{aligned} \tag{10.16}$$

Here the tensor fields indexed in U^δ are like the ones indexed in U in (10.15) (in particular they have a factor $\nabla^{(A)}_{y_1\dots y_A}Y\nabla^{y_A}\upsilon$ with $A\ge 3$) but in addition have rank $\delta>0$ (thus (10.15) is a special case of (10.16) with $\delta=1$). Furthermore, the tensor fields indexed in P are as described in the claim of Lemma 10.6.

Using the generic notation introduced above, we will then show (using generic notation in the first line below) that we can write:

$$
\begin{aligned}
&\sum_{u\in U^{\delta+1}} a_u Xdiv_{i_{\mu+1}}\dots Xdiv_{i_{\mu+\delta+1}}\tilde{C}^{u,i_1\dots i_{\mu+\delta+1}}_g(\Omega_1,\dots,\Omega_p,\phi_\nu,\phi_1,\dots,\hat{\phi}_\nu,\dots,\phi_u)\\
&\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon+\sum_{j\in J} a_j\tilde{C}^j_g(\Omega_1,\dots,\Omega_p,\phi_\nu,\phi_1,\dots,\hat{\phi}_\nu,\dots,\phi_u,\upsilon^\mu)+\\
&\sum_{l\in L^\nu} a_l C^{l,i_1\dots i_\mu}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon+\\
&\sum_{l\in\overline{L}^\nu} a_u X\tilde{d}iv_{i_{\mu+1}} C^{l,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon+\\
&\sum_{p\in P} a_p Xdiv_{i_{\mu+1}} C^{p,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_1,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon=\\
&\sum_{z\in Z} a_z C^z_g(\Omega_1,\dots,\Omega_p,Y,\phi_1,\dots,\phi_u,\upsilon^\mu).
\end{aligned}
\tag{10.17}
$$

If we can prove the above, then by iterative repetition we derive our claim.

Proof of (10.17): We treat the factors $\nabla\upsilon$ as factors $\nabla\phi_h, h>u$ (this can be done easily by a simple polarization). We then notice that (10.16) immediately implies:

$$
\begin{aligned}
&\sum_{u\in U^{\delta}} a_u Xdiv_{i_{\mu+1}}\dots Xdiv_{i_{\mu+\delta}}\tilde{C}^{u,i_1\dots i_{\mu+1}}_g(\Omega_1,\dots,\Omega_p,\phi_\nu,\phi_1,\dots,\hat{\phi}_\nu,\dots,\phi_u)\\
&\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon+\sum_{j\in J} a_j\tilde{C}^j_g(\Omega_1,\dots,\Omega_p,\phi_\nu,\phi_1,\dots,\hat{\phi}_\nu,\dots,\phi_u,\upsilon^\mu)=0,
\end{aligned}
\tag{10.18}
$$

modulo complete contractions of length $\geq\sigma+u+\mu$.

Therefore, we apply our inductive assumption of Corollary 1 in [6] to the above, or if the above falls under a forbidden case of Corollary 1, we then apply the "weak substitute" of that Corollary from the Appenix in [6]. (Notice that if the terms in U^δ contain "forbidden tensor fields" for Corollary 1, then necessarily by construction $\delta>1$). We derive that there exists a linear combination of acceptable tensor fields with a $(u-1+\mu)$-simple character $Ext[\vec{\kappa}_{simp}]$ and with rank $\delta+1$ (indexed in $U^{\delta+1}$ below) so that modulo complete contractions of length $\sigma+u+\mu+\delta$:

$$\sum_{u\in U^\delta} a_u\tilde{C}^{u,i_1\dots i_{\mu+\delta}}_g(\Omega_1,\dots,\Omega_p,\phi_\nu,\phi_1,\dots,\hat{\phi}_\nu,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon$$
$$\nabla_{i_{\mu+1}}\omega\dots\nabla_{i_{\mu+\delta}}\omega - Xdiv_{i_{\mu+\delta+1}}\sum_{u\in U^{\delta+1}} a_u\tilde{C}^{u,i_1\dots i_{\mu+\delta+1}}_g(\Omega_1,\dots,\Omega_p,\phi_\nu,\phi_1,\dots,\hat{\phi}_\nu,\dots,\phi_u)$$
$$\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon\nabla_{i_{\mu+1}}\omega\dots\nabla_{i_{\mu+\delta}}\omega = \sum_{j\in J} a_j C^{u,i_1\dots i_{\mu+\delta}}_g(\Omega_1,\dots,\Omega_p,\phi_\nu,\phi_1,\dots,$$
$$\hat{\phi}_\nu,\dots,\phi_u)\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon\nabla_{i_{\mu+1}}\omega\dots\nabla_{i_{\mu+\delta}}\omega. \tag{10.19}$$

Now, since the above holds formally at the linearized level, it follows that the correction terms of length will be in the form

$$\sum_{p\in P} a_p C^{u,i_1\dots i_{\mu+\delta}}_g\nabla_{i_1}\upsilon\dots\nabla_{i_\mu}\upsilon\nabla_{i_{\mu+1}}\omega\dots\nabla_{i_{\mu+\delta}}\omega + \sum_{j\in J}\dots + \sum_{z\in Z}\dots$$

Then, making the factors $\nabla\omega$ into $Xdiv$'s, by virtue of the last Lemma in the Appendix of [3], we derive our claim. □

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